MATH1010G University Mathematics
Week 4: Limits of Functions

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review

- (one-sided) limits of functions (definition)
- algebraic properties of limits
- relation between limits of sequences and functions
- (one-sided version) sandwich theorem for limits

Limits of functions at infinity

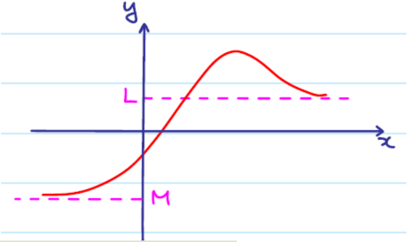
Definition (Informal)

As x tends to $+\infty$ ($-\infty$), if f gets closer and closer to a real number L, then L is called the limit of $f(x)$ at $+\infty$ ($-\infty$), and we write

$$\lim_{x \to +\infty} f(x) = L \quad (\lim_{x \to -\infty} f(x) = L).$$

(sometimes may simply write $\lim_{x \to \infty} f(x)$ instead of $\lim_{x \to +\infty} f(x)$)

From the graph, we have $\lim\limits_{x \to +\infty} f(x) = L$ and $\lim\limits_{x \to -\infty} f(x) = M$



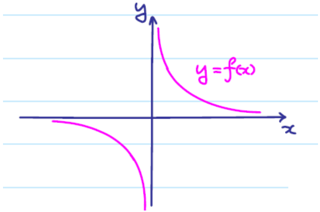
Example

Let $f(x) = \frac{1}{x}, x \neq 0$.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Hence,

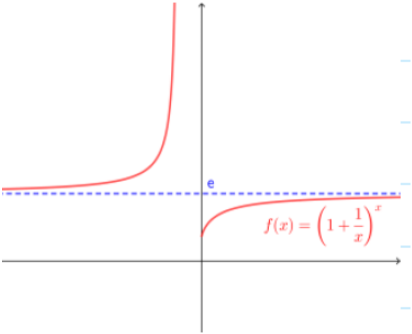
$$\lim_{x\to\pm\infty} f(x) = 0.$$



Theorem

(i) If $k > 0$, then $\lim_{x\to+\infty} \frac{1}{x^k} = 0$.

(ii) $\lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right)^x = e \approx 2.71828$.



Example

Find $\lim_{x \to -\infty} \frac{x}{\sqrt{4x^3+1}}$

compare exponential functions and polynomials.

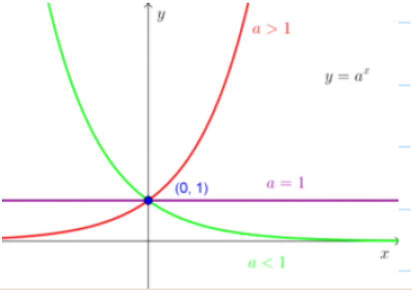
Theorem

- $\lim_{x \to +\infty} x^k e^{-x} = \lim_{x \to +\infty} \frac{x^k}{e^x} = 0$ for any $k > 0$.
- $\lim_{x \to +\infty} p(x) e^{-x} = \lim_{x \to +\infty} \frac{p(x)}{e^x} = 0$ for any polynomial $p(x)$.

message: as x goes to ∞, e^x grows faster than any polynomial.
(proof using L'Hospital's rule)

Theorem

- If $a > 1$, $\lim_{x \to -\infty} a^x = 0$
- If $1 > a > 0$, $\lim_{x \to +\infty} a^x = 0$
- $\lim_{x \to \pm\infty} 1^x = 1$.



Theorem (algebraic properties of limits at infinity)

If both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exist, then

(i) $\lim_{x\to+\infty} f(x) \pm g(x) = \lim_{x\to+\infty} f(x) \pm \lim_{x\to+\infty} g(x)$

(ii) $\lim_{x\to+\infty} f(x) \cdot g(x) = \lim_{x\to+\infty} f(x) \cdot \lim_{x\to+\infty} g(x)$

(iii) $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)}$ if $\lim_{x\to+\infty} g(x) \neq 0$.

Similar results hold for limits at $-\infty$.

the conditions are crucial

Example

Find $\lim_{x\to+\infty}\frac{3x^2}{x^2+x+1}$

Exercise

Find $\lim_{x\to+\infty}\frac{2x+1}{3x^2-2x+1}$ and $\lim_{x\to+\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$.

limits involving e

Example

Find $\lim_{x\to+\infty}\left(1+\frac{1}{2x-1}\right)^{x}$.

Example

Find $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x$.

Example

Find $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$.

Let $y = \frac{1}{x}$, as $x \to 0$, $y \to \pm\infty$. (Not only $+\infty$, but also $-\infty$)

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = \lim_{x\to\pm\infty} \left(1+\frac{1}{y}\right)^{y} = e.$$

Next consider $\lim_{x\to 0} \frac{e^x-1}{x}$.
Idea: when x becomes small (but not zero), both $e^x - 1$ and x are small, but the quotient of them is not small !

Theorem

$\lim_{x\to 0} \frac{e^x-1}{x} = 1.$

Cheating: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} = \lim_{x\to 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots = 1$$

The above is cheating since we are summing up infinitely many small terms, so algebraic properties of limit can not be applied.

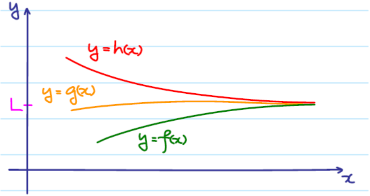
Exercise

Find $\lim\limits_{x\to 0} \frac{e^{3x} - 1}{2x}$.

Theorem (Sandwich Theorem at Infinity)

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions. If $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$ and $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} h(x) = L$, then $\lim_{x\to+\infty} g(x) = L$.

Geometrical meaning:



In fact, the result is still true if $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, +\infty)$

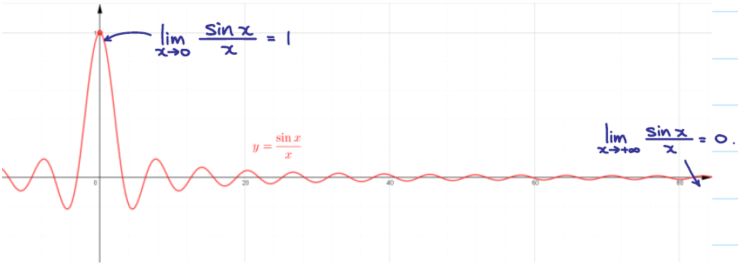
Similar result holds for limits at $-\infty$.

Example

Find $\lim_{x \to +\infty} e^{-x} \sin x$.

Exercise

Show that $\lim_{x \to +\infty} \frac{\sin x}{x} = 0$.



(Don't mix up with $\lim_{x \to 0} \frac{\sin x}{x} = 1$)

Continuity

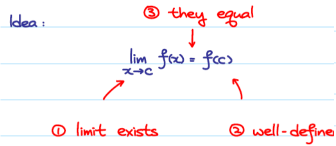
Definition

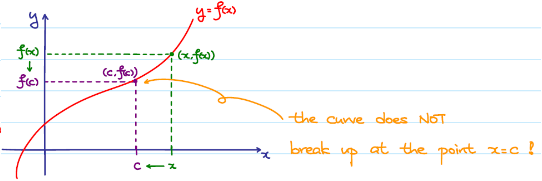
A function f is said to be continuous at $x = c$ if $\lim_{x \to c} f(x)$ exists and is equal to $f(c)$, i.e.,

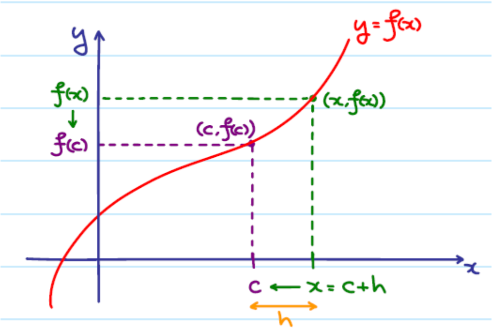
$$\lim_{x \to c} f(x) = f(c)).$$

Further, if f is continuous at every point whenever it is defined, then f is continuous.

Geometrical meaning:







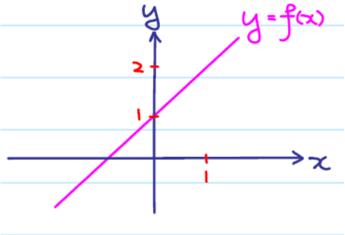
Let $h = x - c$, i.e., $x = c + h$ (Remark: when $x < c$, we have $h < 0$.) When x tends to c, h tends to 0. Thus we may define: A function $f(x)$ is said to be continuous at $x = c$ if

$$\lim_{h \to 0} f(c+h) = f(c).$$

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$.

- $\lim_{x \to 1} f(x) = \lim_{x \to 1} x + 1 = 2$
- $f(1) = 2$
- Hence f is continuous at $x = 1$.



Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ a & \text{if } x = 0. \end{cases}$$

Recall

$$\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

So a function $f(x)$ is said to be continuous at $x = c$ if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

Example

$$\text{If} \quad f(x) = \begin{cases} x^2 - 1 & \text{if} \quad x \geq 1 \\ 1 - x & \text{if} \quad x < 1. \end{cases}$$

Is f continuous at $x = 1$?

Exercise

Show that $f(x) = |x|$ is a continuous function.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

(i) f is continuous at 0.

(ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Show that:

(a) $f(0) = 0$

(b) f is continuous everywhere.

Theorem

(i) If $f(x)$ and $g(x)$ are continuous at $x = c$, then $f(x) \pm g(x)$, $f(x)g(x)$, $\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$ are continuous at $x = c$ as well.

(ii) Polynomial and exponential functions are continuous everywhere.

(iii) Trigonometric and logarithmic functions are continuous at every point where they are defined.

(iv) If $g(x)$ is continuous at $x = c$ and $f(x)$ is continuous at $x = g(c)$, then $f(g(x))$ is continuous at $x = c$.

(That is why we usually have $\lim_{x \to c} f(x) = f(c)$ as we usually looking at continuous functions!)

Example

Let $f(x) = \frac{2x^2+3}{x^2-3x+2}$

quotient of two polynomials (continuous functions)

$$f(x) = \frac{2x^2+3}{(x-2)(x-1)}$$

(the denominator is nonzero when $x \neq 1$ or 2)
Therefore f is continuous in $\mathbb{R} \setminus \{1, 2\}$.

Exercise

$f(x) = ||x| - 1|$

Exercise

Find the values of a and b s.t.

$$f(x) = \begin{cases} 2x - 1, & x < 2, \\ a, & x = 2, \\ x^2 + b, & x > 2 \end{cases}$$

is continuous at $x = 2$

Exercise

The function

$$f = \begin{cases} \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is not continuous at $x = 0$

Sequential criterion for continuity

Recall: $\lim_{x \to c} f(x) = L \Leftrightarrow$

$\forall$ sequence $\{a_n\}$ with $a_n \neq c \quad \forall n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = c$, $\lim_{x\to\infty} f(a_n) = L$.

Theorem

A function f is continuous at $x = c$ if and only if for every sequence $\{a_n\}$ with $a_n \neq c$, $\forall n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = c$, we have

$$\lim_{x\to\infty} f(a_n) = f(\lim_{x\to\infty} a_n) = f(c).$$

Example

Find $\lim_{n\to\infty} \sqrt{\dfrac{n^2+1}{4n^2+3}}$.

Example

Consider

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0. \end{cases}$$

f is not continuous at $x = 0$.

Continuity on $[a, b]$

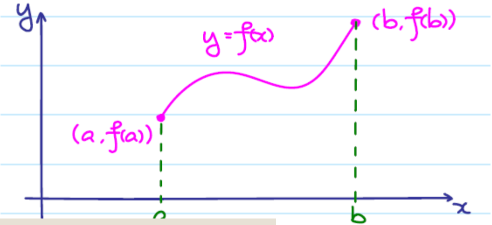
Definition

let $f : [a, b] \to \mathbb{R}$.

- f is said to be continuous at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.
- f is said to be continuous at $x = b$ if $\lim_{x \to b^-} f(x) = f(b)$.

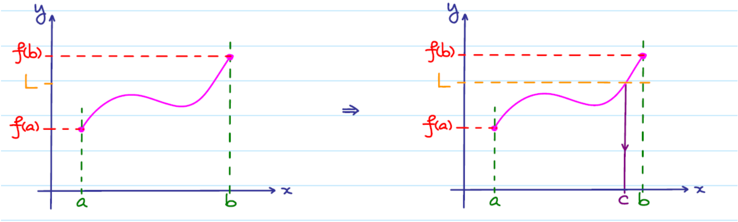
Further if $f : [a, b] \to \mathbb{R}$ is continuous at every point $x \in [a, b]$, then f is said to be continuous on $[a, b]$.

we cannot talk about $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$



intermediate value theorem

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) < f(b)$. Furthermore, if $L \in \mathbb{R}$ such that $f(a) < L < f(b)$, then there exist (at least one) $c \in (a, b)$ such that $f(c) = L$



The similar result holds for $f(a) > L > f(b)$.

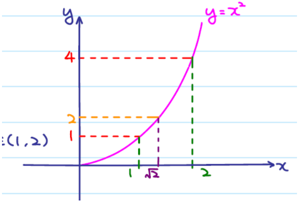
Example

Let $f(x) = x^2$

$$f(1) = 1 < {\color{red}2} < 4 = f(2)$$

f is continuous on $[1, 2]$ (in fact on $\mathbb{R}$)
intermediate value theorem $\Rightarrow$
there exists $c \in (1, 2)$ such that

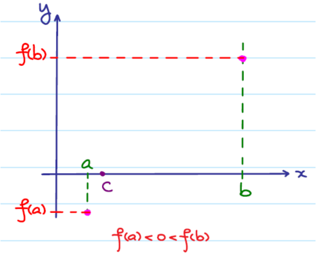
$$f(c) = c^2 = 2.$$



($c = \sqrt{2}$ by definition), and hence $1 < \sqrt{2} < 2$
(a rough approx of $\sqrt{2}$)

Corollary

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. If $f(a)f(b) < 0$ (i.e., $f(a)$ and $f(b)$ have opposite sign) then there exists (at least one) $c \in (a, b)$ such that $f(c) = 0$.



Example

Show that $2^x = \frac{1}{x^2}$ has at least one solution.
(i.e., $f(x) = 2^x - \frac{1}{x^2}$, the equation $f(x) = 0$ has at least one solution.)

Example

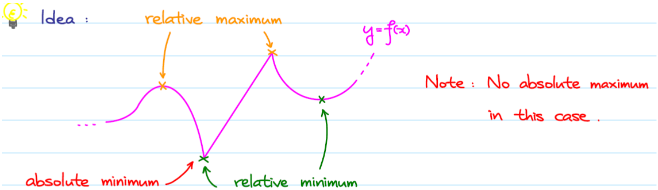
Let $f : [1, 2] \to \mathbb{R}$ is a function defined by $f(x) = x^2 - 2$.

- f is continuous on $[1, 2]$
- $f(1) = -1 < 0$ and $f(2) = 2 > 0$.
- intermediate value theorem: there exists $c \in (1, 2)$ s.t. $f(c) = 0$.
- Since $\sqrt{2}$ is the only root of $f(x) = 0$ on $[1, 2]$, c must be $\sqrt{2}$. Thus, $1 < \sqrt{2} < 2$.

Relative and absolute extrema

Let $f : D \to \mathbb{R}$ be a function.

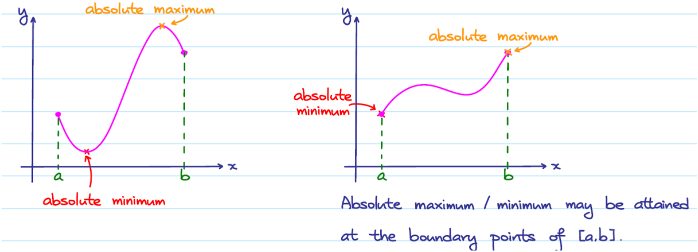
- f has an absolute maxima (minima) at x_0 if $f(x) \leq f(x_0)$ $\big(f(x) \geq f(x_0)\big)$ for all $x \in D$.
- f has a relative maxima (minima) at x_0 if $f(x) \leq f(x_0)$ $\big(f(x) \geq f(x_0)\big)$ for all x in a neighbourhood of x_0 in D.



We simply use maxima/minima to refer to relative maxima/minima,
xima/minima.

Extreme-value theorem (maximum-minimum theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f has an absolute maxima and an absolute minima on $[a, b]$.



Question: Given f, how to find all absolute / relative extrema?
Differentiation provides a powerful tool for that.