MATH1010G University Mathematics
Week 3: Limits of Functions

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Recap: sequence: $\mathbb{Z}^+ \to \mathbb{R}$

- limit of sequence (definition)
- basic facts of limit (constant seq., power a^k, polynomial k^{-a})
- algebraic properties
- sandwich theorem
- monotone convergence theorem

Exercise

Let $a_n = \sum_{j=1}^{n} \frac{1}{n^{3/2}+j}$. What is the limit of the sequence?

Exercise

Consider the recursive sequence: $a_1 = \sqrt{3}$, $a_{n+1} = \sqrt{a_n + 3}$, $n \geq 1$.
Does this sequence have a limit?

Exercise

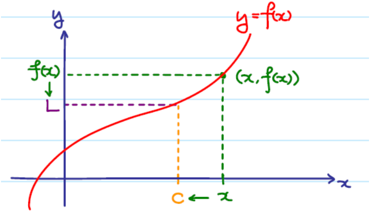
Consider the sequence $a_n = (-1)^n \sin \frac{\pi}{n}$

Definition (informal)

As x gets closer and closer to c from both sides, if $f(x)$ gets closer and closer to $L \in \mathbb{R}$, then L is called the limit of $f(x)$ at c, denoted by

$$\lim_{x \to c} f(x) = L.$$

Also $f(x)$ is said to be convergent to L as x tends to c.

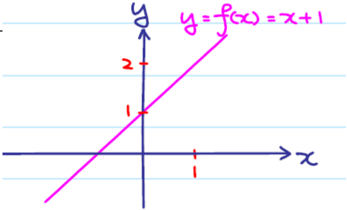


Note: $f(x)$ may not equal to L, even it may be undefined at c!

Example

If $f(x) = x + 1$, find $\lim_{x\to 1} f(x)$.

x	0.9	0.99	0.999	1	1.001
$f(x)$	1.9	1.99	1.999	2	2.001



$f(x)$ tends to 2 as x tends to 1. We write

$$\lim_{x\to 1} f(x) = 2.$$

Example

Let $f(x)$ be a function defined by $f(x) = \frac{x^2-1}{x-1}, \quad x \neq 1.$

We can rewrite f as following:

$$f(x) = \begin{cases} x+1 & \text{if} \quad x \neq 1 \\ \text{undefined} & \text{if} \quad x = 1 \end{cases}$$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	undefined	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1. We write

$$\lim_{x \to 1} f(x) = 2.$$

Example

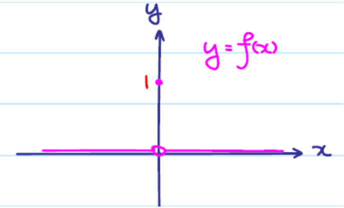
$$f(x) = \begin{cases} 0 & \text{if} \quad x \neq 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0	0	0	1	0	0	0

$$\lim_{x \to 0} f(x) = 0,$$

which does NOT equal to $f(0) = 1$.

The value of f at $x = 0$ is irrelevant !



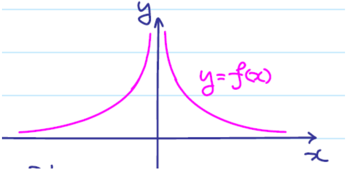
Example

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^2	10^4	10^6	undefined	10^6	10^4	10^2

$f(x)$ tends to $+\infty$ (NOT a real number) as x tends to 0. Therefore

$$\lim_{x \to 0} f(x) \text{ does not exist.}$$



But some still write $\lim_{x \to 0} f(x) = +\infty$ or say $f(x)$ diverges to $+\infty$ as x tends to 0.

Definition

A real number L is called a limit of $f(x)$ at $x = c$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

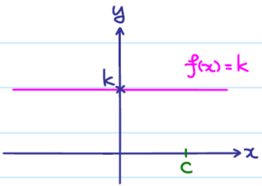
$$|f(x) - L| < \epsilon, \text{whenever } 0 < |x - c| < \delta$$

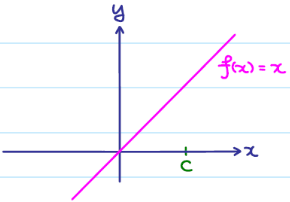
and it is denoted by

$$\lim_{x \to c} f(x) = L$$

Theorem

- *If k is a constant, then*
  $$\lim_{x \to c} k = k$$
  ($f(x) = k$ regarded as a constant function.)
- $\lim_{x \to c} x = c.$





one-sided limit

Definition (Informal)

As x gets closer and closer to c from left (right), if f gets closer and closer to a real number L_{left} (L_{right}), then L_{left} (L_{right}) is called the left (right) hand limit of $f(x)$ at c and we denote it by

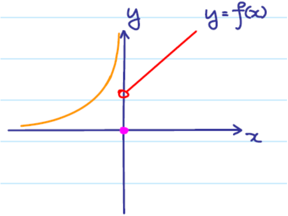
$$\lim_{x\to c^-} f(x) = L_{\text{left}} \; \left(\lim_{x\to c^+} f(x) = L_{\text{right}}\right).$$

Example

$$f(x) = \begin{cases} x+1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ \frac{1}{x^2} & \text{if} \quad x < 0 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+1 = 1;$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2}$$



(does not exist!).

Right hand limit and left hand limit of a function at a point are NOT necessary to be same!

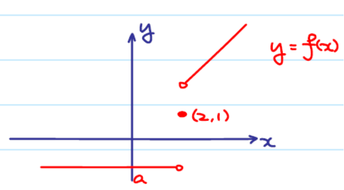
Theorem

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L.$$

Example

If

$$f(x) = \begin{cases} x & \text{if} \quad x > 2 \\ 1 & \text{if} \quad x = 2 \\ a & \text{if} \quad x < 2 \end{cases}$$



Given that $\lim_{x\to 2} f(x)$ exists. What is the value of a?

Example

let $f(x) = |x|$

$$f(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Find $\lim_{x \to 0} f(x)$

Algebraic Properties of Limits

Theorem

If both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then

(i) $\lim_{x \to c} f(x) \pm g(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

(ii) $\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

(iii) $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ if $\lim_{x \to c} g(x) \neq 0$.

Example

Find $\lim_{x \to 2} 3x^2 - 5$.

Example

Find $\lim_{x \to 1} \dfrac{3x^2 - 8}{x - 2}$.

Example

Find $\lim_{x \to 1} \dfrac{x^2 - 1}{x^2 - 3x + 2}$.

Example

let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by $f(x) = \frac{\sqrt{x}-1}{x-1}$. Find $\lim_{x \to 1} f(x)$.

Exercise

Find $\displaystyle\lim_{x \to 4} \frac{x^2 - 16}{\sqrt{x} - 2}$

Exercise

Prove that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Example

Suppose $f(0) = 1$, $g(0) = 2$, $\lim_{x\to 0} \frac{f(x)}{x} = 3$ and $\lim_{x\to 0} \frac{g(x)}{x} = 5$. Find

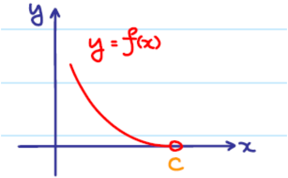
$$\lim_{x\to 0} \frac{f(x)}{g(x)} \text{ and } \lim_{x\to 0} f(x).$$

Theorem

If $f(x) \geq 0$ for $x < c$ ($x > c$) and $\lim_{x\to c^-} f(x)$ $\left(\lim_{x\to c^+} f(x)\right)$ exists, then

$$\lim_{x\to c^-} f(x) \geq 0 \quad \left(\lim_{x\to c^+} f(x) \geq 0\right).$$

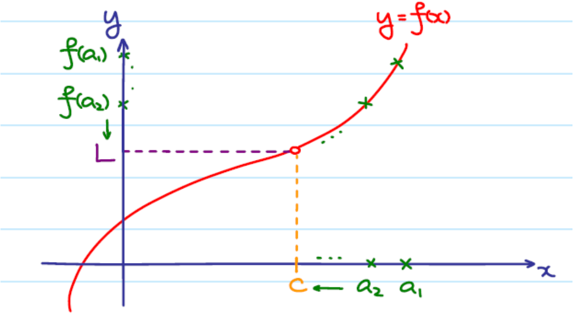
Combine them together, if $f(x) \geq 0$ for $x \neq c$ and $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} f(x) \geq 0$.



relation between limits of sequences and functions

Theorem

$\lim_{x\to c} f(x) = L \Leftrightarrow$ for all sequence $\{a_n\}$ with $a_n \neq c \; \forall n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = c$, we have $\lim_{n\to\infty} f(a_n) = L$.



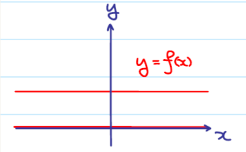
In fact, if we want to show $\lim_{x\to c} f(x) = L$, it is quite impossible to check infinity many sequences. This statement is useful in reverse direction:

- If $\exists\ \{a_n\}$ with $a_n \neq c\ \forall n \in \mathbb{Z}^+$ and $\lim\limits_{n\to\infty} a_n = c$, but $\lim\limits_{n\to\infty} f(a_n)$ does NOT exist, then $\lim_{x\to c} f(x)$ does NOT exist.
- If $\exists\ \{a_n\}, \{b_n\}$ with $a_n, b_n \neq c\ \forall n \in \mathbb{Z}^+$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = c$, but $\lim\limits_{n\to\infty} f(a_n) \neq \lim\limits_{n\to\infty} f(b_n)$, then $\lim_{x\to c} f(x)$ does NOT exist.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$



For any $a \in \mathbb{R}$, f does not have a limit.
It seems that the graph consists of two straight lines, but in fact infinitely many holes are there.

Example

let $f(x) = \sin \frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x \to 0} f(x)$ does NOT exist.

Sandwich theorem

If

$$f(x) \leq g(x) \leq h(x),\ \forall x \in \mathbb{R} \setminus \{c\} \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} g(x) = L.$$

In fact, the result is still true if

$$f(x) \leq g(x) \leq h(x)$$

holds in an open interval containing c but possibly except c.

Geometrical meaning:

Example

Prove that $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$.

Sandwich theorem can be generalize to left and right hand limit. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions.

- If
$$f(x) \le g(x) \le h(x), \forall x < c \text{ and } \lim_{x\to c^-} f(x) = \lim_{x\to c^-} h(x) = L,$$
then
$$\lim_{x\to c^-} g(x) = L.$$
- If
$$f(x) \le g(x) \le h(x),\ \forall x > c \text{ and } \lim_{x\to c^+} f(x) = \lim_{x\to c^+} h(x) = L$$
then
$$\lim_{x\to c^+} g(x) = L.$$

Exercise

Prove that $\lim_{x\to 1^+}(x^2 - 1)\sin(\frac{1}{\sqrt{x-1}}) = 0$.

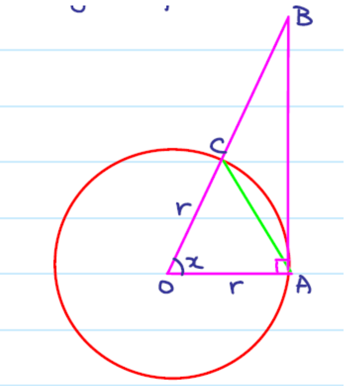
Example

Prove that $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.

Theorem

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

message: When x becomes small (but not zero), both $\sin x$ and x are small, but the quotient of them is not small! (as $x \to 0$, $\sin x$ decays as fast as x.)



Example

Find $\lim_{x\to 0} \dfrac{\sin 3x}{2x}$

Example

Find $\lim_{x\to 0} \dfrac{\cos ax - \cos bx}{x^2}$

Tools:

$$\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

Exercise

Find the limits:

$$\lim_{x \to 0} \frac{\sin 6x - \sin 4x}{\sin 3x - \sin x}, \qquad \lim_{x \to 0} \frac{1 - \cos x}{x \tan x}$$

summary

- (one-sided) limits of functions (definition)
- algebraic properties of limits
- relation between limits of sequences and functions
- (one-sided version) sandwich theorem for limits