MATH1010G University Mathematics
Week 11: Integration (continued)

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brief review on definite integration

- the definition of definite integration
- basic properties of integration (algebraic, comparison, sum)
- mean value theorem for integration
- **fundamental theorem of calculus**

Definite integral using substitution

Theorem

$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$

Example

Evaluate $\int_0^1 8x(x^2+1)dx$.

(similar to indefinite integration)

Example

Evaluate $\int_e^{e^2} \frac{1}{x \ln x}\, dx$.

Definite integration using integration by parts

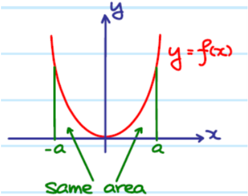
Theorem

$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$

Definite integration of even, odd and periodic functions:

Theorem

If f be even and $a > 0$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.



$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$$

Now,

$$\begin{aligned}
\int_{-a}^{0} f(x)dx &= \int_{a}^{0} -f(y)dy && y = -x, \text{when } x = 0, y = 0 \\
&= \int_{0}^{a} f(y)dy && dy = -dx, x = -a, y = a \\
&= \int_{0}^{a} f(x)dx && \text{(dummy variable)}
\end{aligned}$$

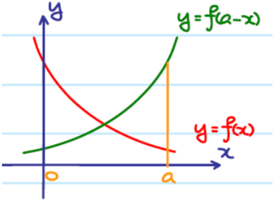
Hence $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
Example: since $|x|$ is an even function,

$$\int_{-4}^{4} |x|dx = 2\int_{0}^{4} |x|dx = 2\int_{0}^{4} xdx = 2[\frac{x^2}{2}]_0^4 = 16.$$

Example

Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\begin{aligned}
&\int_0^a f(a-x)dx \\
&= \int_a^0 -f(y)dy \quad y = a-x, dy = -dx \\
&= \int_0^a f(y)dy \quad \text{when } x = 0, y = a \\
&= \int_0^a f(x)dx \quad x = a, y = 0
\end{aligned}$$



Theorem

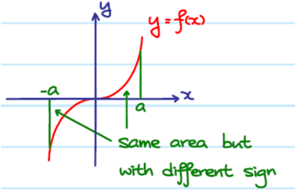
If f is an odd function and $a > 0$, then $\int_{-a}^{a} f(x)dx = 0$.

$$\int_{-a}^{0} f(x)dx$$

$$= \int_{a}^{0} -f(-y)dy \qquad y = -x, dy = -dx$$

$$= \int_{0}^{a} -f(y)dy \qquad \text{when} x = 0, y = 0$$

$$= -\int_{0}^{a} f(x)dx \qquad x = -a, y = a$$



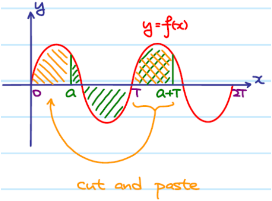
since f is an odd function (i.e., $f(-y) = -f(y)$).

Theorem

If f is periodic with period $T > 0$ and $a \in \mathbb{R}$, show
$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$

$\int_a^{a+T} f(x)dx = \int_a^T f(x)dx + \int_T^{a+T} f(x)dx$

$$\begin{aligned}\int_T^{a+T} f(x)dx &= \int_0^a f(y+T)dy \qquad y = x - T, dy = dx, \\ &= \int_0^a f(y)dy \qquad \text{when} x = T, y = 0 \\ &= \int_0^a f(x)dx \qquad x = a + T, y = a\end{aligned}$$

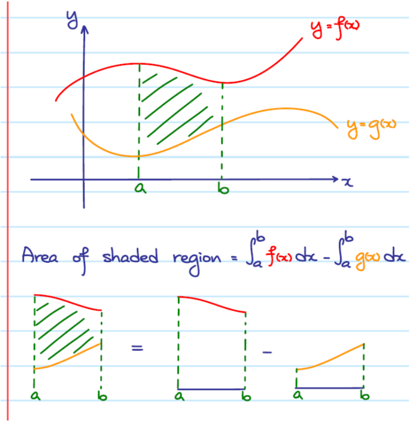


since f is a periodic function with period T (i.e., $f(y+T) = f(y)$)

$$\int_a^{a+T} f(x)dx = \int_a^T f(x)dx + \int_0^a f(x)dx = \int_0^T f(x)dx.$$

area between curves

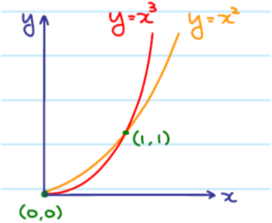
Area of shaded region= $\int_a^b f(x)dx - \int_a^b g(x)dx$



Example

Find the area bounded by $y = x^2$ and $y = x^3$.

Step 1:



$$\text{Solve} \begin{cases} y = x^2 \\ y = x^3 \end{cases} \qquad x^3 = x^2$$

$$\text{i.e., } x^2(x-1) = 0, \quad x = 0 \text{ or } 1$$

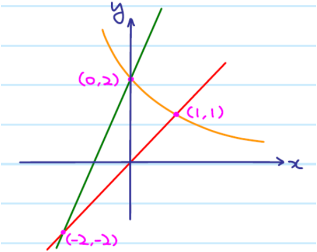
Step 2: Note when $0 \leq x \leq 1, x^3 \leq x^2$

$$\text{area} = \int_0^1 x^2 - x^3 dx = [\frac{x^3}{3} - \frac{x^4}{4}]_0^1 = \frac{1}{12}$$

Example

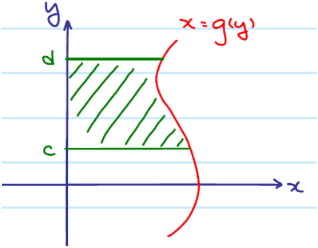
Find the area bounded by $y = f(x) = x$, $y = g(x) = \frac{2}{x+1}$ and $y = h(x) = 2x + 2$.

$$\begin{aligned} \text{area} &= \int_{-2}^{0} (h(x) - f(x))dx \\ &+ \int_{0}^{1} (g(x) - f(x))dx \\ &= \frac{3}{2} + \ln 4 \end{aligned}$$



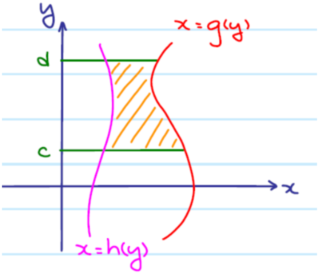
Let $x = g(y)$ and let $c < d$.
The area of the shaded region

$$= \int_c^d g(y)dy$$



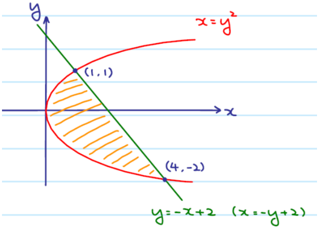
Let $x = g(y)$, $x = h(y)$, and let $c < d$.
The area of the shaded region

$$= \int_c^d (g(y) - h(y))dy$$



Example

The area of the shaded region



$$= \int_{-2}^{1} (-y+2) - y^2 \, dy$$

$$= [-\frac{y^3}{3} - \frac{y^2}{2} + 2y]_{-2}^{1}$$

$$= \frac{9}{2}$$

OR:

$$\int_0^1 \sqrt{x} - (-\sqrt{x}) dx + \int_1^4 (-x+2) - (-\sqrt{x}) dx$$

Improper integrals

Question: Find the area of the unbounded region ?
Idea:

$$\int_a^L f(x)dx$$

area of the unbounded region

$$= \lim_{L\to\infty} \int_a^L f(x)dx$$

(if it exists), denoted by $\int_a^{+\infty} f(x)dx$.

Observation: $\lim_{x\to+\infty} f(x) = 0$ does NOT guarantee $\lim_{L\to+\infty} \int_a^L f(x)dx$ exists.

Example

Find $\int_0^{+\infty} \frac{1}{(x+1)(3x+2)} dx$

Example

Find $\int_0^{+\infty} xe^{-2x} dx$

Exercise

Show that $\int_0^{+\infty} (1+2x)e^{-x} dx = 3$

Definition

If f is a function such that $\lim_{x\to a^+} f(x) = +\infty$ or $-\infty$ ($\lim_{x\to b^-} f(x) = +\infty$ or $-\infty$). Then $\int_a^b f(x)dx$ is said to be an improper integral, and is defined as

$$\lim_{L\to a^+} \int_L^b f(x)dx \quad (\lim_{L\to b^-} \int_a^L f(x)dx)$$

(if it exists.)

Example

Find $\int_0^1 \frac{1}{\sqrt{1-x}}dx$

Note that $\lim_{x\to 1^-} \frac{1}{\sqrt{1-x}} = +\infty$

$$\lim_{L\to 1^-} \int_0^L \frac{1}{\sqrt{1-x}}dx = \lim_{L\to 1^-} [-2\sqrt{1-x}]_0^L = \lim_{L\to 1^-} -2\sqrt{1-L} + 2 = 2$$

Exercise

Find $\int_0^\infty \frac{1}{(1+x)\sqrt{x}}dx$

Hint: $\lim_{x\to 0^+} \frac{1}{(1+x)\sqrt{x}} = +\infty$. Show that both $\lim_{L\to 0^+} \int_L^1 \frac{1}{(1+x)\sqrt{x}}dx$ and $\lim_{M\to\infty} \int_1^M \frac{1}{(1+x)\sqrt{x}}dx$ exist and $\int_0^\infty \frac{1}{(1+x)\sqrt{x}}dx$ equals to the sum of them. (Ans: π.)

Inequalities involving integrals

Example

Define $a_n = \int_0^1 \frac{x^n}{1+x^2}\,dx$ for $n \in \mathbb{Z}^+$. Show $\frac{1}{2(n+1)} \leq a_n \leq \frac{1}{n+1}$, and find $\lim\limits_{n\to\infty} a_n$.

Example

Let $I_n = \frac{1}{n}\int_0^1 \frac{\sin nx}{1+x^2}\,dx$, for $n \in \mathbb{Z}^+$, prove $|I_n| \leq \frac{\pi}{4n}$, hence $\lim\limits_{n\to\infty} I_n = 0$.

summary

- connection between limits of infinite sum and integral
- integration of odd, even, periodic functions
- integration by substitution, integration by parts
- area between curves
- integration on unbounded domains, improper integration