MATH1010G University Mathematics
Week 10: Integration

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idea of reduction formulae: Obtain a formula to reduce the complexity of the integrand.

Example

Let $I_n = \int x^n e^x dx$, $n \in \mathbb{Z}^+$. Prove that $I_n = x^n e^x - nI_{n-1}$, for $n \geq 1$.

$$\begin{aligned} I_n = \int x^n e^x dx &= \int x^n de^x \\ &= x^n e^x - \int e^x dx^n = x^n e^x - \int n e^x x^{n-1} dx \\ &= x^n e^x - nI_{n-1} \end{aligned}$$

The formula $I_n = x^n e^x - nI_{n-1}$ is called a reduction formula.
Note that $I_0 = \int e^x dx = e^x + c$, apply this formula repeatedly

Example

Let $I_n = \int \frac{x^n}{\sqrt{x-1}} dx$, $n \in \mathbb{Z}^+ \cup \{0\}$. Deduce a reduction formula for I_n.

Exercise

Deduce a reduction formula for I_n, where n is a nonnegative integer, if

a) $I_n = \int \frac{1}{x^n\sqrt{x+1}} dx$

b) $I_n = \int \frac{1}{(x^2+1)^n} dx$

c) $I_{n,m} = \int (x-1)^n (x+1)^m dx$, derive the recursion for $I_{n,m}$ in terms of $I_{n+1,m-1}$

Example

Let $I_n = \int \tan^n x dx$, $n \in \mathbb{Z}^+ \cup \{0\}$. Show that $I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$ for $n \geq 2$.

Exercise

Deduce a reduction formula for I_n, $n \in \mathbb{Z}^+ \cup \{0\}$, if

a) $I_n = \int x^n \sin x dx$

b) $I_n = \int x^n \cos x dx$

Definite integration: Riemann sum

Goal: Find the area of the region under the curve $y = f(x)$ over an interval $[a, b]$. However, what is the area of a region with a curved boundary?

Idea: How do we find the area of a leaf? We cover it by a transport grid paper and count the number of squares. To "find" the area of an irregular shape, we do approximation by using squares. Now, we do approximation by using rectangles.

A partition of the interval $[a, b]$ is a finite set $x_0, x_1, \cdots, x_n$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and denote $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \cdots, n$. Then, we choose points, $c_1, c_2, \cdots, c_n$, (partition points) s.t.

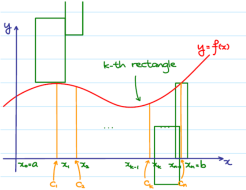
$$x_{k-1} \leq c_k \leq x_k, \quad k = 1, 2, \cdots, n.$$

The interval $[a, b]$ is divided into n sub-intervals. The length of the k-th sub-interval is Δx_k (not necessary the same)

Definition

Let $f : [a, b] \to \mathbb{R}$. The Riemann Sum is defined by $\sum_{k=1}^{n} f(c_k)\Delta x_k$.

- $c_k = x_{k-1}$: the left Riemann sum
- $c_k = x_k$: the right Riemann sum
- $c_k = \frac{x_{k-1}+x_k}{2}$: the mid-point Riemann sum.



The Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$ depends on the function f, partition $a = x_0, x_1, \cdots, x_n = b$ and partition points c_k's chosen.

Example

Let $f(x) = x^2$. Approximate area under $f(x)$ over $[0, 1]$ with 3 even partitions: $0 < \frac{1}{3} < \frac{2}{3} < 1 (x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1)$

Left Riemann Sum: $c_1 = 0, c_2 = \frac{1}{3}, c_3 = \frac{2}{3}$

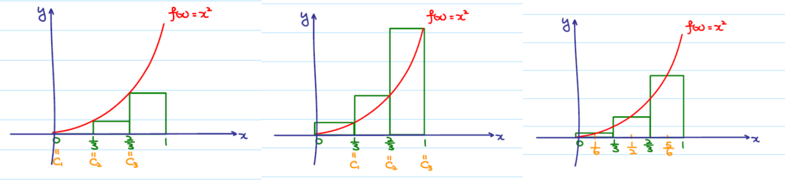
$$\text{area} \approx 0^2 \cdot \frac{1}{3} + (\frac{1}{3})^2 \cdot \frac{1}{3} + (\frac{2}{3})^2 \cdot \frac{1}{3} = \frac{5}{27}$$

Right Riemann Sum: $c_1 = \frac{1}{3}, c_2 = \frac{2}{3}, c_3 = 1$

$$\text{area} \approx (\frac{1}{3})^2 \cdot \frac{1}{3} + (\frac{2}{3})^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} = \frac{14}{27}$$

Mid-point Riemann Sum: $c_1 = \frac{1}{6}, c_2 = \frac{1}{2}, c_3 = \frac{5}{6}$

$$\text{area} \approx (\frac{1}{6})^2 \cdot \frac{1}{3} + (\frac{1}{2})^2 \cdot \frac{1}{3} + (\frac{5}{6})^2 \cdot \frac{1}{3} = \frac{35}{108}$$


y
f(x) = x²
x
0
1/3
2/3
1
c₁
c₂
c₃
1/6
1/2
5/6

Idea: Increasing n (number of rectangles) $\Rightarrow$ better approximation.

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous (or piecewise continuous), and $\Delta x_k = \Delta x = \frac{b-a}{n}$ for $k = 1, 2, \cdots, n$ (even partition), $x_k = a + k\Delta x$ for $k = 0, 1, 2, \cdots, n$, then

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

is independent of the choice of c_k'. The area under $f(x)$ over $[a, b]$ is defined to be this number, denoted by $\int_a^b f(x)dx$.

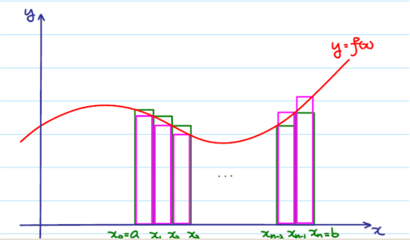
Nothing related to indefinite integration so far !

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1})\Delta x \qquad (\text{take } c_k = x_{k-1})$$

$$= \lim_{n\to\infty} \Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1}) \Big] \Delta x$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \qquad (\text{take } c_k = x_k)$$

$$= \lim_{n\to\infty} \Big[f(x_1) + f(x_2) + \cdots + f(x_n) \Big] \Delta x$$



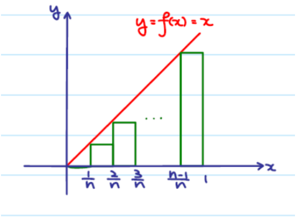
Example

Let $f(x) = x$, for $0 \leq x \leq 1$, take $a = 0$, $b = 1$, $\Delta x = \frac{b-a}{n} = \frac{1}{n}$, $x_k = \frac{k}{n}$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k-1})\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} f(\frac{k-1}{n})\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k-1}{n}\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^2}(\sum_{k=1}^{n} k-1)$$

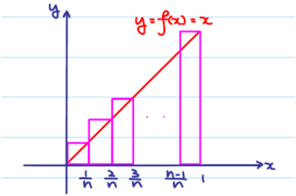
$$= \lim_{n\to\infty}\frac{1}{n^2}\left[\frac{n(n-1)}{2}\right] = \frac{1}{2}$$



$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} f(\frac{k}{n})\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^2}(\sum_{k=1}^{n} k)$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right] = \frac{1}{2}$$



rules for definite integration

Theorem

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that $a \leq b$

(1) If m is a constant, $\int_a^b mf(x)dx = m\int_a^b f(x)dx$

(2) $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

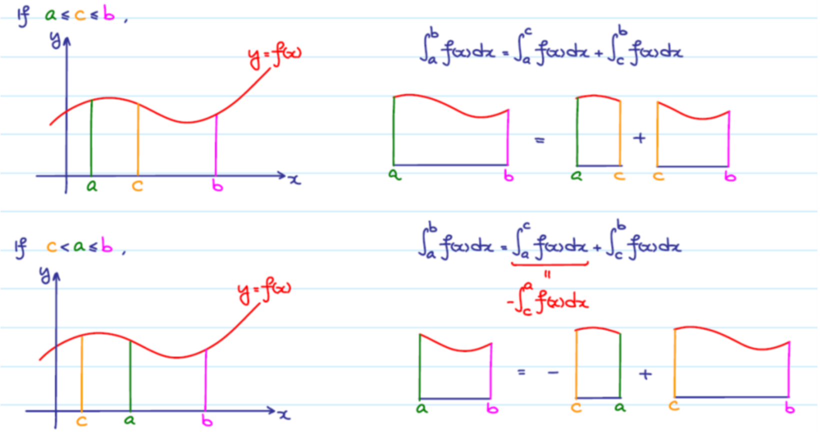
simple facts

(1) $\int_a^a f(x)dx = 0$

(2) $\int_b^a f(x)dx = -\int_a^b f(x)dx$ (reverse direction)

Theorem

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any $c \in \mathbb{R}$ (subdivision)



Theorem

If: $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{\infty} f(c_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\frac{b-a}{n} \geq 0$$

Corollary

If $f, g : [a, b] \to \mathbb{R}$ are continuous functions such that $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

$f(x) - g(x) \geq 0$ on $[a, b]$
$\Rightarrow \int_a^b f(x) - g(x)dx \geq 0 \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

Corollary

If $f : [a, b] \to \mathbb{R}$ is continuous, then

$$\left|\int_a^b f(x)dx\right| \le \int_a^b \left|f(x)\right|dx$$

$$i.e., -\int_a^b \left|f(x)\right|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

Note: $-|f(x)| \le f(x) \le |f(x)|$ for all $x \in [a, b]$, so

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

Corollary

If $f : [a, b] \to \mathbb{R}$ are continuous function, such that $m \le f(x) \le M$ for all $x \in [a, b]$, where $m, M \in \mathbb{R}$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

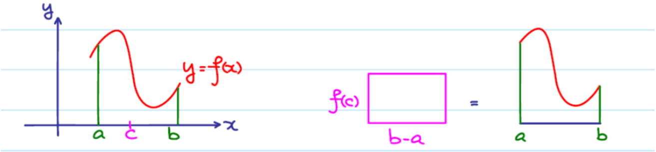
$m \le f(x) \le M$ for all $x \in [a, b]$,

$$\Rightarrow \int_a^b mdx \le \int_a^b f(x)dx \le \int_a^b Mdx \Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

mean value theorem for integrals

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, there exists $c \in [a, b]$ s.t.

$$\int_a^b f(x)dx = f(c)(b - a)$$



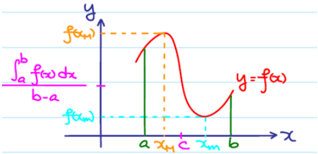
Since f is continuous on $[a, b]$, by the extreme value theorem, there exist $x_m, x_M \in [a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$

Hence,

$$f(x_m)(b-a) \leq \int_a^b f(x)dx \leq f(x_M)(b-a)$$

$$f(x_m) \leq \frac{\int_a^b f(x)dx}{b-a} \leq f(x_M)$$

intermediate value theorem $\Rightarrow$ there exists c that lies between x_m and x_M such that $\int_a^b f(x)dx = f(c)(b-a)$ i.e.,



$$f(c) = \frac{\int_a^b f(x)dx}{b-a}$$

is called the average value of f over $[a, b]$

fundamental theorem of calculus

Let $f(t), t \in \mathbb{R}$, be a continuous function.

- $\int_{x_0}^{x} f(t)dt$ is well defined for all $x \in \mathbb{R}$

Now, we define

$$A(x)\text{=Area under the curve } y = f(t) \text{ over } [x_0, x] = \int_{x_0}^{x} f(t)dt$$

What is the relation between $A(x)$ and $f(x)$?

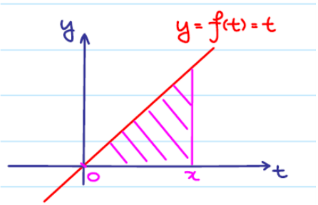
Example

Let $f(t) = t$, $x_0 = 0$

Area of the shaded triangle

$$A(x) = \int_0^x f(t)dt = \frac{1}{2}x^2.$$

Note: $A'(x) = f(x) = x$,
$A(x)$ is an antiderivative of $f(x)$!!!



Fundamental Theorem of Calculus

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $x_0 \in \mathbb{R}$. Let $A(x)$ be defined by

$$A(x) = \int_{x_0}^{x} f(t)dt,$$

Then $A(x)$ is a differentiable function and $A'(x) = f(x)$, i.e. $A(x)$ is an anti-derivative of $f(x)$.

Consequence: If we have an anti-derivative $F(x)$ of $f(x)$, then $A(x) = F(x) + C$

$$\begin{aligned}
\int_a^b f(x)dx &= \int_{x_0}^b f(x)dx - \int_{x_0}^a f(x)dx \\
&= \int_{x_0}^b f(t)dt - \int_{x_0}^a f(t)dt \\
&= A(b) - A(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a)
\end{aligned}$$

That is, if we know an anti-derivative $F(x)$ of $f(x)$, then we can compute the area under the graph of $f(x)$ over $[a, b]$.

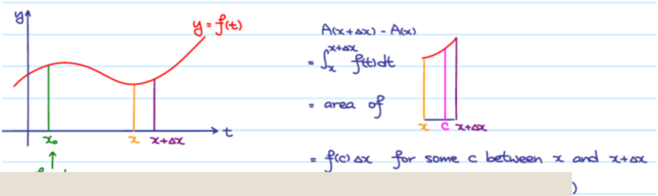
Claim: Let $A(x) = \int_{x_0}^{x} f(t)dt$,

$$\lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = f(x), \quad \text{i.e. } A'(x) = f(x).$$

Now, $A(x+\Delta x) - A(x) = \int_{x}^{x+\Delta x} f(t)dt = f(c)\Delta x$ for some c between x and $x+\Delta x$. (mean value for theorem for integrals)

$$\lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} f(c) = \lim_{c \to x} f(c) = f(x)$$

(as Δx tends to 0, c tends to x), and by continuity of f, $A(x)$ is differentiable and $A'(x) = f(x)$.

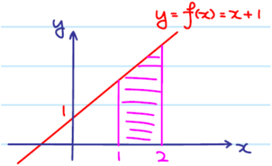


Example

Let $f(x) = x + 1$.

anti-derivative of $f(x)$

$$\int x + 1 dx = \frac{x^2}{2} + x + c$$



Choose $c = 0$, let $F(x) = \frac{x^2}{2} + x$
Area of the shaded region

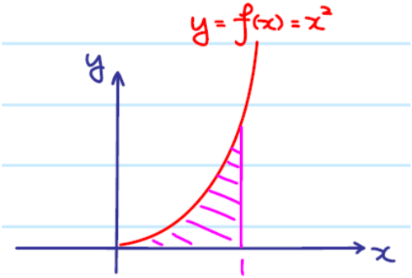
$$= \int_1^2 f(x)dx = F(2) - F(1) = 4 - \frac{3}{2} = \frac{5}{2}$$

Example

Let $f(x) = x^2$

area of the shaded region

$$= \int_0^1 f(x)dx = [\frac{x^3}{3}]_0^1 = (\frac{1^3}{3}) - (\frac{0^3}{3}) = \frac{1}{3}$$

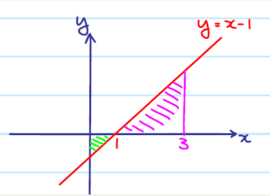


Example (signed area)

$$\int_0^1 x - 1 dx = [\frac{x^2}{2} - x]_0^1 = -\frac{1}{2}$$

$$\int_1^3 x - 1 dx = [\frac{x^2}{2} - x]_1^3 = 2$$

$$\int_0^3 x - 1 dx = [\frac{x^2}{2} - x]_0^3 = \frac{3}{2}$$



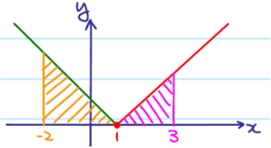
Example

Find $\int_{-2}^{3} |x-1|dx$.

Recall: We can rewrite

$$|x-1| = \begin{cases} x-1 & \text{if} \quad x \geq 1 \\ -(x-1) & \text{if} \quad x < 1 \end{cases}$$

$$\begin{aligned} \int_{-2}^{3} |x-1|dx &= \int_{-2}^{1} |x-1|dx + \int_{1}^{3} |x-1|dx \\ &= \int_{-2}^{1} -(x-1)dx + \int_{1}^{3} x-1\,dx \\ &= \frac{9}{2} + 2 = \frac{13}{2} \end{aligned}$$



Example

Find $\frac{dF}{dx}$ if a) $F(x) = \int_0^x e^{\cos t} dt$, b) $F(x) = \int_0^{x^2} e^{\cos t} dt$, c) $F(x) = \int_x^{x^2} e^{\cos t} dt$

a) $\frac{dF}{dx} = e^{\cos x}$ (directly from Fund. Theorem of Calculus, $f(x) = e^{\cos x}$)
b) by chain rule

$$\frac{dF}{dx} = \frac{d}{dx^2} \int_0^{x^2} e^{\cos t} dt \frac{dx^2}{dx} = e^{cos x^2} \cdot 2x = 2x e^{cos x^2}$$

c)

$$\frac{dF}{dx} = \frac{d}{dx} \int_0^{x^2} e^{\cos t} dt - \frac{d}{dx} \int_0^{x} e^{\cos t} dt = 2x e^{\cos x^2} - e^{\cos x}$$

Example (finding limits by integrals)

Find $\lim_{n\to\infty} \frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \cdots + \frac{n^2}{n^3} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3}$

Note: As $n \to \infty$, it is an infinite sum, summing infinitely many terms, and algebraic rule does NOT work! We cannot say:

$$\lim_{n\to\infty} \frac{1^2}{n^3} = \lim_{n\to\infty} \frac{2^2}{n^3} = \cdots = \lim_{n\to\infty} \frac{n^2}{n^3} = 0$$

and thus

$$\lim_{n\to\infty} \frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \cdots + \frac{n^2}{n^3} = 0$$

idea: Regard the infinite sum as the left or right Riemann sum of some function, so the infinite sum is just the area under that function over an interval. area under $f(x)$ over $[a,b] = \int_a^b f(x)dx$

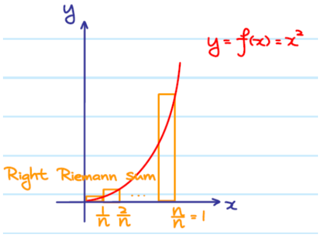
$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x \quad (Left) = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \quad (Right)$$

with $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$
In this case, take $a = 0, b = 1$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^2}\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(\frac{k}{n})\frac{1}{n}$$

$$= \int_0^1 f(x)dx = \int_0^1 x^2 dx = [\frac{1}{3}x^3]_0^1 = \frac{1}{3}$$



Example

Find $\lim_{n\to\infty} \frac{1}{n}(e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^{\frac{n}{n}}) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{\frac{i}{n}}$

Example

Find $\lim_{n\to\infty} \frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+(n-1)^2}$