MATH1010F University Mathematics

Final Review

Gary Choi

November 30, 2023

https://www.math.cuhk.edu.hk/course/2324/math1010f

Final exam

- Date: December 22 (Friday)
- Time: 09:30 - 11:30
- Venue: University Gymnasium
- Closed book, closed notes
- Bring student ID card, black/blue pen
- List of approved calculators:
  `http://www.res.cuhk.edu.hk/images/content/examinations/use-of-calculators-during-course-examination/Use-of-Calculators-during-Course-Examinations.pdf`
- Scope: EVERYTHING!
  - Limits and continuity
  - Differentiation
  - Integration

Basic notations

Set: a collection of elements

- $\{a, b, c\}$ = a set containing three elements a, b, c
- $x \in A$ means "x is an element of the set A"
- $A \subset B$ (also written as $A \subseteq B$) means "A is a subset of B" (i.e. for any element $x \in A$, we have $x \in B$)
- $\{x : \cdots\} = \{x | \cdots\} = \{x \text{ such that } \cdots\}$
- $\mathbb{R}$ = the set of all real numbers
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ = the set of all integers
- $\mathbb{N} = \mathbb{Z}^+ = \{x \in \mathbb{Z} : x > 0\} = \{1, 2, 3, \ldots\}$
  = the set of all positive integers
- $\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{p}{q} \text{ for some } p, q \in \mathbb{Z} \text{ with } q \neq 0\}$
  = the set of all rational numbers
- $\emptyset = \{\ \}$ = empty set

Examples:

- $2 \in \mathbb{Z}$ (since 2 is an integer)
- $\pi \notin \mathbb{Q}$ (since π is an irrational number)
- $\{0, 2, 4, 6, \ldots\} \subset \mathbb{Z}$

Basic notations

- Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- Union of multiple sets $A_1, A_2, \ldots, A_n$:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

- Intersection of multiple sets $A_1, A_2, \ldots, A_n$:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

- Set difference: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

Examples:

- $\{1,2,3\} \cup \{1,3,4,7\} = \{1,2,3,4,7\}$
- $\{1,2,3\} \cap \{1,3,4,7\} = \{1,3\}$
- $\{1,2,3\} \setminus \{1,3,4,7\} = \{2\}$

Basic notations

Intervals:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ (open interval)
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval)
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, \infty) = \{x \in \mathbb{R} : x > a\}$
- $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$
- $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$
- $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

Examples:

- $(-1, 3) \cup (0, 4] = (-1, 4]$
- $[0, 5] \cap (1, \infty) = (1, 5]$
- $(0, 5) \setminus (1, 2) = (0, 1] \cup [2, 5)$
- $\bigcup_{n \in \mathbb{Z}} [2n\pi, (2n+1)\pi) = \cdots \cup [-2\pi, -\pi) \cup [0, \pi) \cup [2\pi, 3\pi) \cup \cdots$

(Lecture 1–2) Sequences

Examples:

- $a_n = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$
- $b_n = 2^{n-1} = 1, 2, 4, 8, \ldots$
- $c_n = (-1)^n = -1, 1, -1, 1, \ldots$
- Arithmetic sequences: $a_{n+1} - a_n = d$ for some constant d
- Geometric sequences: $a_{n+1} = ra_n$ for some constant r

Definitions:

- Monotonic increasing (or "increasing"): $a_n \leq a_{n+1}$ for all n
- Monotonic decreasing (or "decreasing"): $a_n \geq a_{n+1}$ for all n
- Monotonic: Either monotonic increasing or decreasing
- Strictly increasing: $a_n < a_{n+1}$ for all n
- Strictly decreasing: $a_n > a_{n+1}$ for all n
- Bounded below: there exists $M \in \mathbb{R}$ s.t. $a_n > M$ for all n
- Bounded above: there exists $M \in \mathbb{R}$ s.t. $a_n < M$ for all n
- Bounded: there exists $M \in \mathbb{R}$ s.t. $|a_n| < M$ for all n
  (i.e. both bounded below and bounded above)

(Lecture 1–2) Limits of sequences

Definitions:

- (Convergent sequence) If $\{a_n\}$ approaches a number L as n approaches infinity, we say $\lim_{n\to\infty} a_n = L$.
- (Divergent sequence) If no such L exists, we say that $\{a_n\}$ is divergent.
  Note: If $\lim_{n\to\infty} a_n = \infty$ or $-\infty$, it is also divergent.

Uniqueness of limit: If a_n is convergent, then the limit is unique.

Basic arithmetic rules: If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

- $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$
- $\lim_{n\to\infty} (ca_n) = ca$ (where c is a constant)
- $\lim_{n\to\infty} a_n b_n = ab$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ (if $b \neq 0$)

Example: $\lim_{n\to\infty} \left(\cos\frac{1}{n} - 2\left(\frac{3}{4}\right)^n + \frac{1}{n^2} \right) = 1 - 2\cdot 0 + 0 = 1$

(Lecture 1–2) Limits of sequences

Limits involving $\pm\infty$:

- $\infty \pm L = \infty$
- $-\infty \pm L = -\infty$
- $\infty + \infty = \infty$
- $-\infty - \infty = -\infty$
- $L \cdot \infty = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases}$
- $\frac{L}{\pm\infty} = 0$
- (Indeterminate forms) $\infty - \infty$, $\frac{\pm\infty}{\pm\infty}$, $\frac{0}{0}$, $0 \cdot \infty$: try further simplifying

Convergence $\Rightarrow$ **Boundedness**:

If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded.

Remark: The converse is **NOT** true, i.e. bounded $\not\Rightarrow$ convergent!
Example: $\{(-1)^n\} = -1, 1, -1, 1, \ldots$ is bounded but divergent.

(Lecture 2) Monotone convergence theorem

If $\{a_n\}$ is monotonic and bounded, then $\{a_n\}$ is convergent.

Other versions:

- If $\{a_n\}$ is monotonic increasing and bounded above, then $\{a_n\}$ is convergent.
- If $\{a_n\}$ is monotonic decreasing and bounded below, then $\{a_n\}$ is convergent.

Example: To prove that $\{a_n\}$ with $\begin{cases} a_{n+1} = \sqrt{a_n + 1} \\ a_1 = 1 \end{cases}$ is convergent, we prove that (i) $\{a_n\}$ is bounded by 2 (by MI) and (ii) $\{a_n\}$ is monotonic increasing.

Remark:
The converse is **NOT** true: convergent $\not\Rightarrow$ monotonic & bounded!
Example:
$\{\frac{(-1)^n}{n}\} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$ converges to 0, but the sequence is not monotonic.

(Lecture 3) Squeeze theorem (sandwich theorem)

If $b_n \le a_n \le c_n$ for all n and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} a_n = L$.

Example: $\lim_{n\to\infty} \frac{\sin(\cos n)}{n} = ?$

Solution: Since $-1 \le \sin(\cos n) \le 1$ for all n, we have

$$\frac{-1}{n} \le \frac{\sin(\cos n)}{n} \le \frac{1}{n}.$$

Now, since $\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$, by squeeze theorem, we have

$$\lim_{n\to\infty} \frac{\sin(\cos n)}{n} = 0.$$

Some possible ways to show that a sequence converges

(I) **Find the limit directly using some basic limit results**

- $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$, $\lim_{n\to\infty} \frac{1}{n} = 0$, ...

Example: $\lim_{n\to\infty}\left(\cos\frac{1}{n} + \left(\frac{3}{4}\right)^n + \frac{1}{n^2}\right) = 1 + 0 + 0 = 1$

(II) **Use the monotone convergence theorem**

- Show that the sequence is bounded and monotonic (may need to use mathematical induction)
- Conclude that the sequence converges (i.e. can write $\lim_{n\to\infty} a_n = L$, then solve some equations to find L if needed).

Example: Show that $\begin{cases} a_{n+1} = \sqrt{a_n + 1} \\ a_1 = 1 \end{cases}$ converges.

(III) **Use the squeeze theorem**

- Find b_n, c_n s.t. $b_n \le a_n \le c_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n \ (= L)$.
- Conclude that $\lim_{n\to\infty} a_n = L$.

Example: Show that $\{a_n\} = \left\{\frac{(-1)^n + \sin n}{n}\right\}$ converges.

If a way does not work, it **does NOT imply** that the sequence is divergent! Try another way.

Some possible ways to show that a sequence diverges

(I) **Show that $\{a_n\}$ is unbounded** (i.e. $\lim_{n\to\infty} |a_n| = \infty$)

- Reason: If a sequence converges, it must be bounded

Example: $a_n = (-1)^n n^2$ diverges as $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n^2 = \infty$

(II) **Show that $\{a_n\}$ contains two subsequences which converge to two different values**

- Reason: If a sequence converges, then the limit must be unique

Example: $a_n = (-1)^n$ diverges since $\{a_1, a_3, a_5, \dots\}$ converges to -1 and $\{a_2, a_4, a_6, \dots\}$ converges to 1.

If a way does not work, it **does NOT imply** that the sequence is convergent! Try another way.

(Lecture 3) Infinite series

Series: $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$

Examples:

- $\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- (Arithmetic sum) $\sum_{k=1}^{n} (a + (k-1)d) = \frac{2a + (n-1)d}{2}$
- (Geometric sum) $\sum_{k=1}^{n} ar^{k-1} = \frac{a(r^n - 1)}{(r-1)}$ (if $r \neq 1$)

Convergence of infinite series: We say that an infinite series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ is convergent if the sequence of partial sums $\{s_n\}$ (where $s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$) converges.

Example: (Euler's number) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718$

(Lecture 3) Functions

Definitions:

- $f : A \to B$
  - A: Domain
  - B: Codomain
  - f: Some rule of assigning elements in A to elements in B
- Range of $f = \{f(x) : x \in A\}$ (also known as image of f)
- Natural domain = largest domain on which f can be defined

Examples:

- For $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, the range of f is $[0, \infty)$
- The natural domain of $f(x) = \frac{1}{\sqrt{x+1}}$ is $(-1, \infty)$
- The natural domain of $\tan(x)$ is

$$\mathbb{R} \setminus \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots\} = \bigcup_{n\in\mathbb{Z}} \left((n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi\right)$$

(Lecture 3–4) Injective, subjective, bijective functions, and inverse functions

- $f : A \to B$ is said to be injective (or "1-1", "one-to-one") if for any $x_1, x_2 \in A$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$
(Or equivalently, if $f(x_1) = f(x_2)$ then we have $x_1 = x_2$)
- $f : A \to B$ is said to be surjective (or "onto") if for any $y \in B$, there exists $x \in A$ such that $y = f(x)$
- f is bijective if it is both injective and surjective
- If $f : A \to B$ is a bijective function, the inverse function $f^{-1} : B \to A$ satisfies $f^{-1}(f(x)) = x$ for all $x \in A$ and $f(f^{-1}(y)) = y$ for all $y \in B$

Examples:

- $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$ is bijective
- $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is not injective as $f(-1) = f(1) = 1$
- $f : [0, \infty) \to \mathbb{R}$ with $f(x) = x^2$ is injective but not surjective
- $f : [0, \infty) \to [0, \infty)$ with $f(x) = x^2$ is bijective, and the inverse function is $f^{-1} : [0, \infty) \to [0, \infty)$ with $f^{-1}(y) = \sqrt{y}$

(Lecture 3–4) Even, odd, periodic functions

- f is an even function if $f(-x) = f(x)$ for all x
- f is an odd function if $f(-x) = -f(x)$ for all x
- f is a periodic function if there exists a constant k such that $f(x) = f(x+k)$ for all x

Examples:

- $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$ for all x
- $f(x) = x^3 + \sin x$ is odd because
  $f(-x) = (-x)^3 + \sin(-x) = -x^3 - \sin x = -(f(x))$ for all x
- $f(x) = x + 1$ is neither odd nor even because $f(-1) = 0 \neq \pm f(1)$
- $f(x) = 3\sin x + \cos\frac{x}{2}$ is periodic because
  $f(x+4\pi) = 3\sin(x+4\pi) + \cos\frac{x+4\pi}{2} =$
  $3\sin(x+4\pi) + \cos\left(\frac{x}{2} + 2\pi\right) = 3\sin x + \cos\frac{x}{2} = f(x)$ for all x

(Lecture 4–5) Some common functions

Exponential function $e^x : \mathbb{R} \to \mathbb{R}^+$

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- bijective function

Logarithmic function $\ln : \mathbb{R}^+ \to \mathbb{R}$

- Inverse function of e^x ($y = e^x \Leftrightarrow x = \ln y$)
- bijective function

Sine function $\sin : \mathbb{R} \to [-1, 1]$

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- odd function (because $\sin(-x) = -\sin x$)
- periodic function (because $\sin(x + 2\pi) = \sin x$)

Cosine function $\cos : \mathbb{R} \to [-1, 1]$

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$
- even function (because $\cos(-x) = \cos x$)
- periodic function (because $\cos(x + 2\pi) = \cos x$)

(Lecture 4–5) Limit of functions

Definitions:

- Left-hand limit: We say that $\lim_{x \to a^-} f(x) = L$ if $f(x)$ is close enough to L whenever x is close enough to a and $x < a$.
- Right-hand limit: We say that $\lim_{x \to a^+} f(x) = L$ if $f(x)$ is close enough to L whenever x is close enough to a and $x > a$.
- Two-sided limit: We say that $\lim_{x \to a} f(x) = L$ if both the left-hand limit and the right-hand limit exist and are equal, i.e.

$$\boxed{\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L}$$

Remark: Whether f is defined at a or the value of f at a is **NOT** important for finding $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, $\lim_{x \to a} f(x)$

Example: If $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x^2 & \text{if } x > 0 \end{cases}$, we have

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$,
so the two-sided limit exists and we have $\lim_{x \to 0} f(x) = 0$ $(\neq 1)$

(Lecture 4–5) Properties of limits of functions

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

- $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$ (where c is a constant)
- $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (if $\lim_{x\to a} g(x) \neq 0$)

Examples:

- $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right) = \lim_{x\to 0} \frac{(x+1)-1}{x(x+1)} = \lim_{x\to 0} \frac{1}{x+1} = 1$
- $\lim_{x\to 2} \frac{2-x}{3-\sqrt{x^2+5}} = \lim_{x\to 2} \left(\frac{2-x}{3-\sqrt{x^2+5}} \cdot \frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}\right)$

  $= \lim_{x\to 2} \frac{(2-x)(3+\sqrt{x^2+5})}{4-x^2} = \lim_{x\to 2} \frac{3+\sqrt{x^2+5}}{2+x} = \frac{6}{4} = \frac{3}{2}$

(Lecture 4–5) Properties of limits of functions

Some other useful limit results:

- $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$
- $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$
- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

Examples:

- $\lim\limits_{x\to 0} \dfrac{e^{3x} - 1}{x} = \lim\limits_{x\to 0} \dfrac{e^{3x} - 1}{3x} \cdot 3 = 1 \cdot 3 = 3$
- $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x} = \lim\limits_{x\to 0} \dfrac{\frac{\sin 2x}{2x}(2x)}{\frac{\sin 3x}{3x}(3x)} = \dfrac{\left(\lim\limits_{x\to 0} \dfrac{\sin 2x}{2x}\right) \cdot 2}{\left(\lim\limits_{x\to 0} \dfrac{\sin 3x}{3x}\right) \cdot 3} = \dfrac{1 \cdot 2}{1 \cdot 3} = \dfrac{2}{3}$

(Lecture 5) Sequential criterion

We have $\lim_{x\to a} f(x) = L$ (limit of function)

if and only if

For any sequence $\{x_n\}$ with $x_n \neq a$ for any n and $\lim_{n\to\infty} x_n = a$,

we have $\lim_{n\to\infty} f(x_n) = L$ (limit of sequence).

Consequence: If we can find two sequences $\{x_n\}, \{y_n\}$ such that:

- $x_n \neq a$, $y_n \neq a$ for all n and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$
- but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$,

then $\lim_{x\to a} f(x)$ does not exist.

Example: Prove that $\lim_{x\to 0} \sin \dfrac{1}{x}$ does not exist.

Solution: Let $\{x_n\} = \left\{\frac{1}{n\pi}\right\} = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \cdots$ and

$\{y_n\} = \left\{\frac{1}{2n\pi+\frac{\pi}{2}}\right\} = \frac{1}{2\pi+\frac{\pi}{2}}, \frac{1}{4\pi+\frac{\pi}{2}}, \frac{1}{6\pi+\frac{\pi}{2}}, \cdots$, then we have

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ but $\lim_{n\to\infty} f(x_n) = 0 \neq \lim_{n\to\infty} f(y_n) = 1$.

(Lecture 5) Squeeze theorem for functions

Let f, g, h be functions. If $f(x) \leq g(x) \leq h(x)$ for any $x \neq a$ on a neighborhood of a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then the limit of $g(x)$ at $x = a$ exists and we have $\lim_{x \to a} g(x) = L$.

Example: $\lim_{x \to 0} x \sin \frac{1}{e^{x^2} - 1} = ?$

Solution:

Since $-1 \leq \sin \frac{1}{e^{x^2}-1} \leq 1$ for all x, we have $-x \leq x \sin \frac{1}{e^{x^2} - 1} \leq x$.

As $\lim_{x \to 0}(-x) = 0 = \lim_{x \to 0} x$, by squeeze theorem, $\lim_{x \to 0} x \sin \frac{1}{e^{x^2} - 1} = 0$.

(Lecture 6–7) Limits at infinity

Definitions:

- We say that $\lim_{x\to\infty} f(x) = L$ if $f(x)$ is close enough to L whenever x is large enough.
- (Similar for $\lim_{x\to-\infty} f(x)$)

Examples:

- $\lim_{x\to\infty} \frac{1}{x-1} = 0$
- $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$
- $\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{3x} = \lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{3x\cdot\frac{2}{2}} =$
  $\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{2x\cdot\frac{3}{2}} = \left(\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{2x}\right)^{\frac{3}{2}} = e^{\frac{3}{2}}$
- $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$ and $\lim_{x\to\infty} \frac{(\ln x)^k}{x} = 0$ for any positive integer k

(Lecture 7) Continuity of functions

f is said to be continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

In other words, we have:

(i) The limit $\lim_{x \to a} f(x)$ exists (i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$), **and**

(ii) It is equal to the value of f at $x = a$.

f is said to be continuous on an interval (a, b) if f is continuous at every point on (a, b).

Examples:

- x^n, $\cos x$, $\sin x$, e^x are continuous on $\mathbb{R}$
- $\ln(x)$ is continuous on $\mathbb{R}^+$
- $f(x) = \begin{cases} -x + 1 & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$

(Lecture 7) Properties of continuous functions

Properties:

- If $f(x)$ and $g(x)$ are continuous at $x = a$, then the following functions are also continuous at $x = a$:
  - $f(x) \pm g(x)$
  - $cf(x)$ (where c is a constant)
  - $f(x)g(x)$
  - $\frac{f(x)}{g(x)}$ (if $g(a) \neq 0$)
- If $f(x)$ is continuous at $x = a$ and $g(u)$ is continuous at $u = f(a)$, then the composition $(g \circ f)(x)$ (i.e. $g(f(x))$) is also continuous at $x = a$.

Examples:

- $\cos(x) + 2x$ is continuous on $\mathbb{R}$ because both $\cos x$ and x are continuous on $\mathbb{R}$.
- $\sin(x^3 + 1)$ is continuous at $x = 0$ because $x^3 + 1$ is continuous at $x = 0$ and $\sin(u)$ is continuous at $u = 1$.

(Lecture 7) Intermediate value theorem and extreme value theorem

Intermediate value theorem (IVT):

> Let f be a continuous function on $[a, b]$.
> For any real number L between $f(a)$ and $f(b)$
> (i.e. $f(a) < L < f(b)$ or $f(b) < L < f(a)$),
> there exists $c \in (a, b)$ such that $f(c) = L$.

Example: Show that $f(x) = x^7 + x^3 + 1$ has a real root.

Solution: Note that $f(-1) = -1 < 0$ and $f(0) = 1 > 0$. As f is continuous, by IVT, there exists $c \in (-1, 0)$ s.t. $f(c) = 0$.

Extreme value theorem (EVT):

> Let f be a continuous function on $[a, b]$. Then there exists
> $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$
> (i.e. f has a global maximum and a global minimum in $[a, b]$).

(Lecture 8) Differentiability of functions

f is said to be differentiable at $x = a$ if the following limit (called the derivative of f at $x = a$) exists:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Another form:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Remark: For piecewise functions, we need to check both

$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$

Example (finding derivative by definition, i.e. **first principle**):

- If $f(x) = x^2$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x.$$

(Lecture 8–9) Derivatives of polynomial, exponential, logarithmic, and trigonometric functions

- $(x^n)' = nx^{n-1}$
- $(e^x)' = e^x$
- $(\ln x)' = \dfrac{1}{x}$
- $(a^x)' = a^x \ln a$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x = \dfrac{1}{\cos^2 x}$
- $(c)' = 0$ (where c is a constant)
- $(\sinh x)' = \cosh x$ (where $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$)
- $(\cosh x)' = \sinh x$
- $(\tanh x)' = \text{sech}^2 x = \dfrac{1}{\cosh^2 x}$

(Lecture 8–9) Differentiation rules (sum, difference, product, and quotient rules)

If f and g are differentiable at a point, then the following functions are also differentiable at that point:

- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(cf(x))' = cf'(x)$ (where c is a constant)
- Product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

- Quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad \text{(if } g(x) \neq 0\text{)}$$

Examples:

- $(x^3 \sin x)' = (x^3)' \sin x + x^3(\sin x)' = 3x^2 \sin x + x^3 \cos x$
- $\left(\frac{\sin x}{x^2+1}\right)' = \frac{(\sin x)'(x^2+1)+(\sin x)(x^2+1)'}{(x^2+1)^2} = \frac{(x^2+1)\cos x + 2x \sin x}{(x^2+1)^2}$

(Lecture 8–9) Differentiation rules (chain rule)

Chain rule:

If $f(x)$ is differentiable at $x = a$ and $g(u)$ is differentiable at $u = f(a)$, then $(g \circ f)$ is differentiable at $x = a$ and we have

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

In other words, we have

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Examples:

- $(\sin x^2)' = \frac{d(\sin u)}{du}\frac{du}{dx}$ (let $u = x^2$) $= (\cos u)(2x) = 2x \cos x^2$
- $(e^{\sin x})' = e^{\sin x} \cos x$

A more complicated version: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

Example:

- $\left(\ln(\cos(x^3))\right)' = \dfrac{1}{\cos x^3} \cdot (-\sin(x^3)) \cdot (3x^2) = -3x^2 \tan x^3$

(Lecture 8–9) Continuity and differentiability

Property:

If f is differentiable at $x = a$, then f is continuous at $x = a$

The converse is **NOT** true: if f is continuous at $x = a$, it may or may not be differentiable at $x = a$

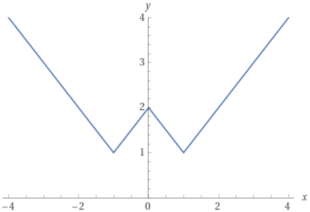
Example: $f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

- $f(x)$ is continuous on $\mathbb{R}$ (i.e. at every point $x \in \mathbb{R}$):
  - For any $a < 0$, $\lim_{x \to a} f(x) = \lim_{x \to a} (-x) = -a = f(a)$
  - For any $a > 0$, $\lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a)$
  - For $a = 0$, we have $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0 = f(0)$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 = f(0)$, and hence $\lim_{x \to 0} f(x) = f(0)$
- $f(x)$ is not differentiable at $x = 0$:
  Note that $f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h \to 0} \dfrac{|h| - 0}{h} = \lim_{h \to 0} \dfrac{|h|}{h}$ but $\lim_{h \to 0^+} \dfrac{|h|}{h} = \lim_{h \to 0^+} \dfrac{h}{h} = 1$ and $\lim_{h \to 0^-} \dfrac{|h|}{h} = \lim_{h \to 0^+} \dfrac{-h}{h} = -1$

(Lecture 8–9) Continuity and differentiability

Another example of continuous but not differentiable functions:

$$f(x) = |x+1| - |x| + |x-1|$$
$$= \begin{cases} -(x+1) - (-x) - (x-1) & = -x & \text{if } x < -1 \\ (x+1) - (-x) - (x-1) & = x+2 & \text{if } -1 \leq x < 0 \\ (x+1) - (x) - (x-1) & = -x+2 & \text{if } 0 \leq x < 1 \\ (x+1) - (x) + (x-1) & = x & \text{if } x \geq 1 \end{cases}$$



- $f(x)$ is continuous on $\mathbb{R}$
- $f(x)$ is not differentiable at $x = -1, 0, 1$

(Lecture 10–11) Implicit differentiation

Idea: Find y' without having to explicitly write $y = f(x)$.

Example:
If $x \sin y + y^2 = x + 3y$, find the slope of tangent at $(0, 0)$.

Solution:

$$\begin{aligned} (x \sin y + y^2)' &= (x + 3y)' \\ (\sin y + x(\cos y)y') + 2yy' &= 1 + 3y' \\ (x \cos y + 2y - 3)y' &= 1 - \sin y \\ y' &= \frac{1 - \sin y}{x \cos y + 2y - 3} \end{aligned}$$

The slope of tangent at $(0, 0)$ is $\dfrac{1 - \sin 0}{0 \cdot \cos 0 + 2 \cdot 0 - 3} = -\dfrac{1}{3}$

(Lecture 10–11) Logarithmic differentiation

Idea: Find the derivative of some complicated functions using logarithms.

Example: If $y = x^x$, find y'.

Solution:

$$
\begin{aligned}
y &= x^x \\
\ln y &= \ln(x^x) \\
\ln y &= x \ln x \\
(\ln y)' &= (x \ln x)' \\
\frac{1}{y} y' &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\
y' &= y(\ln x + 1) = x^x(\ln x + 1)
\end{aligned}
$$

(Lecture 10–11) Derivatives of inverse functions

Inverse functions:
If $f(y)$ is a bijective and differentiable function with $f'(y) \neq 0$ for any y, then the inverse function $y = f^{-1}(x)$ is differentiable:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Examples:

$$y = \sin^{-1} x \Rightarrow \sin y = x \Rightarrow (\cos y)y' = 1 \Rightarrow (\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$$

$$y = \cos^{-1} x \Rightarrow \cos y = x \Rightarrow (-\sin y)y' = 1 \Rightarrow (\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow (\sec^2 y)y' = 1 \Rightarrow (\tan^{-1} x)' = \frac{1}{1+x^2}$$

(Lecture 11–12) Higher order derivatives

- **Second derivative**:

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

- n-**th derivative**:

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\frac{dy}{dx}\right)\right)\right)$$

- 0-**th derivative**:

$$y^{(0)} = f^{(0)}(x) = f(x)$$

Examples:

- $(\sin x^2)'' = ((\sin x^2)')' = ((\cos x^2)(2x))'$
  $= (-\sin x^2)(2x)(2x) + 2\cos x^2 = -4x^2 \sin x^2 + 2\cos x^2$
- Find y'' if $xy + \sin y = 1$:

$$(xy + \sin y)' = 1' \Rightarrow (y + xy' + y'\cos y) = 0 \Rightarrow y' = \frac{-y}{x + \cos y}$$

$$\Rightarrow y'' = -\frac{y'(x + \cos y) - y(1 - y'\sin y)}{(x + \cos y)^2} = \frac{2y(x + \cos y) + y^2 \sin y}{(x + \cos y)^3}$$

(Lecture 11–12) Higher order differentiation rules

If f and g are n-times differentiable (i.e. $f^{(n)}$ and $g^{(n)}$ exist), then:

- $(f \pm g)^{(n)} = f^{(n)} \pm g^{(n)}$
- $(cf)^{(n)} = cf^{(n)}$ (where c is a constant)
- Leibniz's rule (product rule for higher order derivatives):

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

where $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$ is the binomial coefficient.

Example: $(x^3 \sin x)^{(4)}$

$= 1 \cdot (x^3)'''' \sin x + 4 \cdot (x^3)'''(\sin x)' + 6 \cdot (x^3)''(\sin x)'' + 4(x^3)'(\sin x)'''$
$\quad +1 \cdot x^3 (\sin x)''''$
$= 0 + 24 \cos x - 36x \sin x - 12x^2 \cos x + x^3 \sin x$
$= (x^3 - 36x) \sin x + (24 - 12x^2) \cos x$

(Lecture 12–13) n-times differentiability and continuity

If f is n-times differentiable at $x = a$
($f^{(n)}(a)$ exists, i.e. $f^{(n-1)}$ is differentiable at $x = a$),
then $f^{(n-1)}$ is continuous at $x = a$.

$$
\begin{gathered}
f \text{ is } n\text{-times differentiable at } x = a \text{ (i.e. } f^{(n)}(a) \text{ exists)} \\
\Downarrow \\
f^{(n-1)}(a) \text{ exists and } f^{(n-1)} \text{ is continuous at } x = a \\
\Downarrow \\
\vdots \\
\Downarrow \\
f'(a) \text{ exists and } f' \text{ is continuous at } x = a \\
\Downarrow \\
f \text{ is continuous at } x = a
\end{gathered}
$$

However, the converse is **NOT** true!

Example: Let $f(x) = |x|x$, then:

- f is differentiable at $x = 0$
- f' is continuous at $x = 0$
- but f' is not differentiable at $x = 0$ (i.e. $f''(0)$ does not exist)

(Lecture 14) Local extrema, critical points, turning points

Local maximum:

$f(x)$ has a local maximum at $x = a$ if $f(x) \leq f(a)$ for all x near a (more precisely, for all $x \in D \cap (a - \delta, a + \delta)$ where D is the domain and $\delta > 0$ is some small number).

Local minimum:

$f(x)$ has a local minimum at $x = a$ if $f(x) \geq f(a)$ for all x near a.

Note:

Local extremum points can be either interior points or endpoints!

Example: For $f : [-\pi, \pi] \to \mathbb{R}$ with $f(x) = \sin x$,
local maximum points $= (-\pi, 0)$, $(\frac{\pi}{2}, 1)$
local minimum points $= (-\frac{\pi}{2}, -1)$, $(\pi, 0)$.

Critical points:

f has a critical point at $x = a$ if $f'(a) = 0$ or $f'(a)$ does not exist.

Turning points:

f has a turning point at $x = a$ if f' changes sign at a.

Note: $\{\text{Turning points}\} \subset \{\text{Critical points}\}$

Example: $x = 0$ is a critical point of $f(x) = x^3$, but it is not a turning point.

(Lecture 14) First and second derivative tests

Theorem:

Let $f(x)$ be a continuous function. If $f(x)$ has a local maximum/ minimum at $x = a$, then $x = a$ must be a critical point of $f(x)$.

First derivative test:

Let $f(x)$ be a continuous function and $x = a$ be a critical point.

(i) If f' changes sign from $+$ to $-$ at a, then $f(x)$ has a local maximum at $x = a$.

(ii) If f' changes sign from $-$ to $+$ at a,then $f(x)$ has a local minimum at $x = a$.

Second derivative test:

Let $f(x)$ be a continuous function.

(i) If $f'(a) = 0$ and $f''(a) < 0$, then $f(x)$ has a local maximum at $x = a$.

(ii) If $f'(a) = 0$ and $f''(a) > 0$, then $f(x)$ has a local minimum at $x = a$.

(Lecture 15) Finding global extrema

Extreme value theorem (EVT) for closed and bounded intervals:

> Let f be a continuous function on $[a, b]$. Then there exists $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$ (i.e. f has a global maximum and a global minimum in $[a, b]$).

Note: For f on (a, b), $(a, b]$, or $[a, b)$, f may NOT have any global extrema in some cases!

Finding global extrema for functions on general intervals:

1. Check all critical points (including endpoints if applicable) to find all local extrema.
2. Compare the values of $f(x)$ at all such points as well as the limit of f as x approaches the open endpoints (if applicable) to determine the existence of global extrema.

Examples:

$f(x) = x^2$ on $[-2, 1]$: global min. point $= (0, 0)$; global max. $= (-2, 4)$

$f(x) = x^2$ on $\mathbb{R}$: global minimum point $= (0, 0)$; no global max.

$f(x) = x^2$ on $(0, 1)$: no global min; no global max

(Lecture 15) Concavity and points of inflection

Concavity:
We say that $f(x)$ is

- concave upward on (a, b) if $f''(x) > 0$ on (a, b)
- concave downward on (a, b) if $f''(x) < 0$ on (a, b)

Example: $f(x) = x^3 \implies f''(x) = 6x$
f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$

Point of inflection:
We say that $x = a$ is an inflection point of $f(x)$ if $f''(x)$ changes sign at $x = a$.
Example: $f(x) = x^3 \implies f''(x) = 6x$
As f'' changes sign from $-$ to $+$ at $x = 0$, f has an inflection point at $x = 0$.

(Lecture 15) Asymptotes (vertical, horizontal, oblique)

Vertical asymptotes:

- $x = a$ is a vertical asymptote of $f(x)$ if

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

Example: For $f(x) = x^2 + \frac{1}{x-1}$,
$x = 1$ is a vertical asymptote since $\lim_{x \to 1^+} f(x) = \infty$.

Horizontal asymptotes:

- $y = b$ is a horizontal asymptote of $f(x)$ if

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b$$

Note: $f(x)$ can have at most two different horizontal asymptotes (one for $\lim_{x \to -\infty}$ and one for $\lim_{x \to \infty}$)

Example: For $f(x) = e^x$,
$y = 0$ is a horizontal asymptote since $\lim_{x \to -\infty} f(x) = 0$.

(Lecture 15) Asymptotes (vertical, horizontal, oblique)

Oblique asymptotes:

- $y = ax + b$ is an oblique asymptote of $f(x)$ if

$$\lim_{x \to -\infty} (f(x) - (ax + b)) = 0 \quad \text{or} \quad \lim_{x \to \infty} (f(x) - (ax + b)) = 0$$

- Note: $f(x)$ can have at most two different oblique asymptotes (one for $\lim_{x \to -\infty}$ and one for $\lim_{x \to \infty}$)

Example: For $f(x) = x + 3 + \frac{2}{x}$, $y = x + 3$ is an oblique asymptote since $\lim_{x \to \infty} (f(x) - (x + 3)) = \lim_{x \to \infty} \frac{2}{x} = 0$.

- Finding oblique asymptotes:
  Method 1: Directly work on $f(x) - (ax + b)$, then check the coefficients of different terms and see what a, b have to be such that the limit $= 0$ as $x \to \infty$ or $-\infty$.
  Method 2: Find a such that $a = \lim_{x \to \infty} \frac{f(x)}{x}$ (or $\lim_{x \to -\infty}$), then find $b = \lim_{x \to \infty} (f(x) - ax)$ (or $\lim_{x \to -\infty}$).

(Lecture 15) Asymptotes (vertical, horizontal, oblique)

Example: $f(x) = \sqrt{x^2 - 2x + 3}$

- No vertical asymptote (as $f(x)$ is defined everywhere on $\mathbb{R}$)
- No horizontal asymptote ($\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$)
- Oblique asymptotes:

For $x \to \infty$, we have

$$a = \lim_{x\to\infty} \frac{\sqrt{x^2 - 2x + 3}}{x} = \lim_{x\to\infty} \sqrt{1 - \frac{2}{x} + \frac{3}{x^2}} = 1, \text{ and}$$

$$b = \lim_{x\to\infty} (\sqrt{x^2 - 2x + 3} - x) = \lim_{x\to\infty} \frac{(x^2 - 2x + 3) - x^2}{\sqrt{x^2 - 2x + 3} + x} = -1$$

For $x \to -\infty$, we have

$$a = \lim_{x\to-\infty} \frac{\sqrt{x^2 - 2x + 3}}{x} = \lim_{x\to-\infty} -\sqrt{1 + \frac{2}{x} + \frac{3}{x^2}} = -1, \text{ and}$$

$$b = \lim_{x\to-\infty} (\sqrt{x^2 - 2x + 3} + x) = \lim_{x\to-\infty} \frac{(x^2 - 2x + 3) - x^2}{\sqrt{x^2 - 2x + 3} - x} = 1$$

So the oblique asymptotes are $y = x - 1$ and $y = -x + 1$.

(Lecture 15) Curve sketching

To sketch a given function, do the following:

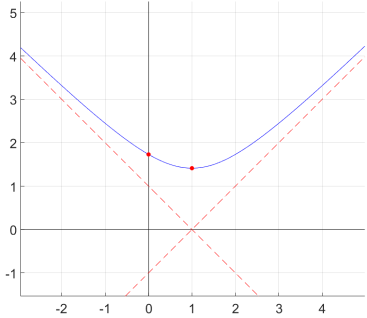
1. Find:
   - (Natural) domain
   - x-intercept
   - y-intercept
   - Asymptotes (vertical, horizontal, oblique)
   - Critical points (and check whether they are local max/min)
   - Inflection points (and check concavity)
2. Sketch the curve based on the information above.

Examples: See the main MATH1010 lecture notes.

(Lecture 15) Curve sketching

Example: $f(x) = \sqrt{x^2 - 2x + 3}$

- Domain: $\mathbb{R}$ (as $\sqrt{x^2 - 2x + 3} = \sqrt{(x-1)^2 + 2}$ is defined everywhere)
- x-intercept: None (as $f(x) = \sqrt{(x-1)^2 + 2} \neq 0$)
- y-intercept: $f(0) = \sqrt{3}$
- Asymptotes: $y = x - 1$ and $y = -x + 1$ (see the previous slide)
- Critical points: $f'(x) = \frac{x-1}{\sqrt{x^2-2x+3}}$, so the only critical point is at $x = 1$.
  By first derivative test, it is a local minimum.
- Inflection point: None (as $f''(x) = \frac{2}{\sqrt{x^2-2x+3}} > 0$)



(Lecture 15–17) Mean value theorem (MVT)

Rolle's theorem:

If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Lagrange's mean value theorem:

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Cauchy's mean value theorem:

If f, g are continuous on $[a, b]$, differentiable on (a, b), with $g(a) \neq g(b)$ and $g'(x) \neq 0$ on (a, b), then there exists $c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

Using MVTs to prove inequalities:

Example: Prove that $|\cos(x) - \cos(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Solution:

- If $x = y$, we have $|\cos(x) - \cos(y)| = 0 = |x - y|$.
- If $x \neq y$, by Lagrange's MVT, there exists c between x and y such that
$$\frac{\cos(x) - \cos(y)}{x - y} = -\sin(c).$$
Therefore, we have
$$\frac{|\cos(x) - \cos(y)|}{|x - y|} = |-\sin(c)| \leq 1 \iff |\cos(x) - \cos(y)| \leq |x - y|$$
for all $x, y \in \mathbb{R}$.

(Lecture 16) Derivatives and inequalities

Increasing/decreasing functions and derivatives:

- f is (monotonic) increasing on (a, b) (i.e. $f(x) \leq f(y)$ for all $x, y \in (a, b)$ with $x < y$) if and only if $f'(x) \geq 0$ on (a, b).
- f is (monotonic) decreasing on (a, b) (i.e. $f(x) \geq f(y)$ for all $x, y \in (a, b)$ with $x < y$) if and only if $f'(x) \leq 0$ on (a, b).
- f is constant on (a, b) if and only if $f'(x) = 0$ on (a, b).
- f is strictly increasing on (a, b) (i.e. $f(x) < f(y)$ for all $x, y \in (a, b)$ with $x < y$) **if** $f'(x) > 0$ on (a, b).
- f is strictly decreasing on (a, b) (i.e. $f(x) > f(y)$ for all $x, y \in (a, b)$ with $x < y$) **if** $f'(x) < 0$ on (a, b).

Using derivatives to prove inequalities:

Example: Let $p > 1$. Prove that $(1 + x)^p > 1 + px$ for all $x > 0$.

Solution: Let $f(x) = (1 + x)^p - (1 + px)$. Then

$$f'(x) = p(1 + x)^{p-1} - p > 0$$

for all $x > 0$. Therefore, f is strictly increasing on $(0, \infty)$. We have

$$f(x) > f(0) = 0 \Longrightarrow (1 + x)^p > 1 + px.$$

(Lecture 17) L'Hopital's rule

L'Hopital's rule:

Let $a \in \mathbb{R}$ or $a = \pm\infty$. If f and g are differentiable near a and all of the following conditions are satisfied:

1. Both $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ or both $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$.
2. $g'(x) \neq 0$ near a.
3. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or $= \pm\infty$.

Then we have $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Remarks:

- Similar results hold for one-sided limit ($\lim_{x \to a^-}$ and $\lim_{x \to a^+}$)
- Sometimes may need to apply the rule more than once
- Not always applicable! Check if the requirements are satisfied.

(Lecture 17) L'Hopital's rule

Handling different indeterminate forms:

- $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$: May try to apply the L'Hopital's rule directly

  Example:

  $$\lim_{x\to 0}\frac{\tan x - x}{x^3}\ \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\ \left(\frac{0}{0}\right)$$

  $$= \lim_{x\to 0}\frac{2\sec x\sec x\tan x}{6x} = \lim_{x\to 0}\frac{\sin x}{3x\cos^3 x} = \frac{1}{3}$$

- $0\cdot(\pm\infty)$, $\infty-\infty$: May try to convert them into $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then apply the L'Hopital's rule

  Example:

  $$\lim_{x\to 1}(x^2-1)\tan\frac{\pi x}{2}\quad (0\cdot\infty) = \lim_{x\to 1}\frac{x^2-1}{\cot\frac{\pi x}{2}}\quad \left(\frac{0}{0}\right)$$

  $$= \lim_{x\to 1}\frac{2x}{\frac{\pi}{2}\cdot\csc^2\frac{\pi x}{2}} = \lim_{x\to 1}\frac{2x\sin^2\frac{\pi x}{2}}{\frac{\pi}{2}} = \frac{2\cdot 1\cdot 1^2}{\frac{\pi}{2}} = \frac{4}{\pi}$$

(Lecture 17) L'Hopital's rule

Handling different indeterminate forms:

- 1^{∞}, ∞^{0}, 0^{0}: May use logarithm and apply the L'Hopital's rule to the logged expression, then use $\lim_{x\to a} y = e^{\lim_{x\to a}\ln y}$

  Example: Find $\lim_{x\to 0^+}(x+\sin x)^x$ (0^0)

  Solution: Let $y=(x+\sin x)^x$, then $\ln y = x\ln(x+\sin x)$ and

$$\lim_{x\to 0^+} x\ln(x+\sin x) \quad (0\cdot(\pm\infty)) = \lim_{x\to 0^+}\frac{\ln(x+\sin x)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{x+\sin x}(1+\cos x)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+}\frac{-x(1+\cos x)}{1+\frac{\sin x}{x}}$$

$$= \frac{-0(1+1)}{1+1} = 0$$

  So $\lim_{x\to 0^+}(x+\sin x)^x = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$

(Lecture 18) Taylor polynomial

Taylor polynomial:

The n-th order Taylor polynomial of $f(x)$ about a point $x = a$ is

$$p_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Property: We have $f^{(k)}(a) = p_n^{(k)}(a)$ for all $k = 0, 1, \ldots, n$.

Example:

The 2nd order Taylor polynomial of $f(x) = \sqrt{1+x}$ about $x = 0$ is

$$p_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 = 1 + \frac{x}{2} - \frac{x^2}{8}$$

Taylor's theorem:

Let $x \neq a$ (i.e. $x > a$ or $x < a$).
Suppose $f^{(n)}$ exists and is continuous on $[a, x]$ (or $[x, a]$), and $f^{(n+1)}$ exists on (a, x) (or (x, a)).
Then there exists $c \in (a, x)$ (or (x, a)) such that

$$f(x) = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

(Lecture 19–20) Taylor series

Taylor series:

The Taylor series of $f(x)$ about a point $x = a$ is the infinite series

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Property: If the remainder term in Taylor's theorem $R_n(x) \to 0$ as $n \to \infty$ on an interval I, then the Taylor series is equal to the function (i.e. $f(x) = T(x)$) on I.

Examples: $e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all $x \in \mathbb{R}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ for all $x \in \mathbb{R}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ for all $x \in \mathbb{R}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$ for $|x| < 1$

(Lecture 19–20) Taylor series

Properties:

- If $T(x)$ is the Taylor series of $f(x)$ about $x = 0$, then $T(x^k)$ is the Taylor series of $f(x^k)$ about $x = 0$ for all positive integer k

  Example: The Taylor series of $\frac{\sin x^2}{x^2}$ about 0 is

  $$\frac{1}{x^2}\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right) = 1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \cdots$$

- Addition and subtraction of Taylor series

  Example: The Taylor series of $\frac{\sin x^2}{x^2} + \cos x$ about 0 is

  $$\left(1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \cdots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 2 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

- Multiplication and division of Taylor series

  Example: The Taylor series of $\frac{\sin x^2}{x^2} \cos^3 x$ about 0 is

  $$\left(1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^3 = 1 - \frac{3x^2}{2} + \frac{17x^4}{24} + \cdots$$

(Lecture 19–20) Taylor series

Properties:

- Composition of Taylor series
  Example:
  The Taylor series of $\cos(\sin x)$ about 0 is
  $$1 - \frac{1}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \frac{1}{4!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^4 - \cdots$$
  $$= 1 - \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$$

- Differentiation of Taylor series
  Example:
  The Taylor series of $-\frac{x}{(1+x)^2} = x\left(\frac{1}{1+x}\right)'$ is
  $$x\left(1 - x + x^2 - x^3 - \cdots\right)'$$
  $$= x(-1 + 2x - 3x^2 + \cdots)$$
  $$= -x + 2x^2 - 3x^3 + \cdots$$

(Lecture 20) Using Taylor series to find limits

Idea: To find $\lim_{x\to c} f(x)$, replace certain components in $f(x)$ with their Taylor series (if those components are equal to their Taylor series for x near c)

Example:

$$\lim_{x\to 0} \frac{\ln(1+x) - x\sqrt{1-x}}{x - \sin x}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \mathcal{O}(x^4)\right) - x\left(1 - \frac{x}{2} - \frac{x^2}{8} + \mathcal{O}(x^3)\right)}{x - \left(x - \frac{x^3}{6} + \mathcal{O}(x^5)\right)}$$

$$= \lim_{x\to 0} \frac{\frac{11}{24}x^3 + \mathcal{O}(x^4)}{\frac{1}{6}x^3 + \mathcal{O}(x^5)}$$

$$= \lim_{x\to 0} \frac{\frac{11}{24} + \mathcal{O}(x)}{\frac{1}{6} + \mathcal{O}(x^2)} = \frac{\frac{11}{24} + 0}{\frac{1}{6} + 0} = \frac{11}{4}$$

(Lecture 20) Indefinite integration

Indefinite integral:

Let $f(x)$ be continuous. An antiderivative (or primitive function) of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. The collection of all antiderivatives of $f(x)$ is called the indefinite integral of $f(x)$ and is denoted by $\displaystyle\int f(x)dx$.

We have $\displaystyle\int f(x)dx = F(x) + C$, where C is a constant.

Example: x^2, $x^2 + 3$, $x^2 - 1$ are antiderivatives of $2x$.
More generally, we have $\int 2x\ dx = x^2 + C$.

Properties:

- $$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$
- $$\int kf(x)dx = k\int f(x)dx$$

Example: $\int (x^3 + 2x - 1)\ dx = \frac{x^4}{4} + x^2 - x + C$

(Lecture 20) Some basic integrals

- $\int k \; dx = kx + C$
- $\int x^n \; dx = \dfrac{x^{n+1}}{n+1} + C$ (where $n \neq -1$)
- $\int e^x \; dx = e^x + C$
- $\int \dfrac{1}{x} \; dx = \ln|x| + C$
- $\int a^x \; dx = \dfrac{1}{\ln a} a^x + C$
- $\int \sin x \; dx = -\cos x + C$
- $\int \cos x \; dx = \sin x + C$
- $\int \tan x \; dx = -\ln|\cos x| + C$
- $\int \sec x \; dx =$ $\ln|\sec x + \tan x| + C$
- $\int \sec^2 x \; dx = \tan x + C$
- $\int \sec x \tan x \; dx = \sec x + C$
- $\int \csc x \cot x \; dx = -\csc x + C$
- $\int \dfrac{1}{\sqrt{1-x^2}} \; dx = \sin^{-1} x + C$
- $\int \dfrac{1}{1+x^2} \; dx = \tan^{-1} x + C$

(Lecture 20) Integration by substitution

$$\boxed{\int f(u(x))\frac{du}{dx}dx = \int f(u)du}$$

Example: $\int \sqrt{3x+4}\ dx = ?$

Solution: Let $u = 3x+4$, then $\frac{du}{dx} = 3 \Rightarrow du = 3dx$. We have

$$\int \sqrt{3x+4}\ dx = \int \sqrt{u}\cdot\frac{1}{3}du = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(3x+4)^{3/2} + C$$

Example: $\int e^{2x^2+1}x\ dx = ?$

Solution: Let $u = 2x^2+1$, then $\frac{du}{dx} = 4x \Rightarrow du = 4xdx$. We have

$$\int e^{2x^2+1}x\ dx = \frac{1}{4}\int e^u\ du = \frac{1}{4}e^u + C = \frac{1}{4}e^{2x^2+1} + C$$

Example: $\int \cos x \sin x\ dx = \int \sin x\ d(\sin x) = \frac{\sin^2 x}{2} + C$

(Lecture 21) Trigonometric integrals

Useful trigonometric identities for handling trigonometric integrals:

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
- $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
- $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
- $\cos x \cos y = \dfrac{1}{2}(\cos(x+y) + \cos(x-y))$
- $\cos x \sin y = \dfrac{1}{2}(\sin(x+y) - \sin(x-y))$
- $\sin x \sin y = \dfrac{1}{2}(\cos(x-y) - \cos(x+y))$

Example: $\displaystyle\int \sin^2 x \; dx = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

Example:
$$\int \sin 5x \cos 3x \; dx = \int \frac{1}{2}(\sin 8x + \sin 2x) dx = -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$$

(Lecture 21) Trigonometric integrals

For $\displaystyle\int \cos^m x \sin^n x \; dx$:

- If m is odd, let $u = \sin x$
  Example: $\int \cos^3 x \sin^4 x \; dx = \int \cos^2 x \sin^4 x \cos x \; dx$
  $= \int (1-u^2)u^4 \; du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$
- If n is odd, let $u = \cos x$
  Example: $\int \sin^5 x \; dx = \int \sin^4 x \sin x \; dx = -\int (1-u^2)^2 \; du$
  $= -\int (1 - 2u^2 + u^4) \; du = -u + \frac{2u^3}{3} - \frac{u^5}{5} + C =$
  $-\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + C$
- If both m, n are even, use double angle formulas to reduce the power and then use the above methods (if applicable)
  Example: $\int \sin^4 x \cos^2 x \; dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \cdot \frac{1+\cos 2x}{2} \; dx = \cdots$

For $\displaystyle\int \sec^m x \tan^n x \; dx$:

- If m is even, let $u = \tan x$
- If n is odd, let $u = \sec x$
- If m is odd and n is even, use $\tan^2 x = \sec^2 x - 1$ to write everything in terms of $\sec x$ and use reduction formula

(Lecture 21–22) Trigonometric substitution

Idea: Simplify some integrals (without any trigonometric functions) by substituting $x =$ some trigonometric functions

- For $\sqrt{a^2 - x^2}$, substitute $x = a\sin\theta$
- For $\sqrt{a^2 + x^2}$, substitute $x = a\tan\theta$
- For $\sqrt{x^2 - a^2}$, substitute $x = a\sec\theta$

Example: $\int \frac{1}{\sqrt{9-x^2}}\, dx = ?$

Solution: Let $x = 3\sin\theta$, then $dx = 3\cos\theta\ d\theta$. We have

$$\int \frac{1}{\sqrt{9-x^2}}\, dx = \int \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}} d\theta = \int 1\ d\theta = \theta + C = \sin^{-1}\frac{x}{3} + C$$

Example: $\int \frac{x^3}{\sqrt{1+x^2}}\, dx = ?$

Solution: Let $x = \tan\theta$, then $dx = \sec^2\theta\ d\theta$. We have $\int \frac{x^3}{\sqrt{1+x^2}}\ dx$
$= \int \frac{\tan^3\theta\sec^2\theta}{\sqrt{1+\tan^2\theta}} d\theta = \int \tan^3\theta\sec\theta\ d\theta = \int(\sec^2\theta - 1)\ d(\sec\theta)$
$= \frac{\sec^3\theta}{3} - \sec\theta + C = \frac{(\sqrt{1+x^2})^3}{3} - \sqrt{1+x^2} + C$

Example with different possible substitutions

<u>Example:</u> $\displaystyle\int \frac{x^3}{(x^2+1)^3}\ dx = ?$

<u>Method 1</u>: Let $u = x^2 + 1$. We have $du = 2x\ dx$ and so

$$\int \frac{x^3}{(x^2+1)^3}\ dx = \frac{1}{2}\int \frac{x^2}{(x^2+1)^3}\ 2x\ dx = \frac{1}{2}\int \frac{u-1}{u^3}\ du$$

$$= \frac{1}{2}\left(-\frac{1}{u} + \frac{1}{2u^2}\right) + C = -\frac{1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C = -\frac{2x^2+1}{4(x^2+1)^2} + C$$

<u>Method 2</u>: Let $x = \tan\theta$. We have $dx = \sec^2\theta\ d\theta$ and so

$$\int \frac{x^3}{(x^2+1)^3}\ dx = \int \frac{\tan^3\theta}{(\tan^2\theta+1)^3}\sec^2\theta\ d\theta = \int \frac{\tan^3\theta}{\sec^6\theta}\sec^2\theta\ d\theta$$

$$= \int \sin^3\theta\cos\theta\ d\theta = \int \sin^3\theta\ d(\sin\theta) = \frac{\sin^4\theta}{4} + C = \frac{1}{4\csc^4\theta} + C$$

$$= \frac{1}{4(1+\cot^2\theta)^2} + C = \frac{1}{4(1+\frac{1}{x^2})^2} + C = \frac{x^4}{4(x^2+1)^2} + C$$

Note: The results are consistent as $\frac{x^4}{4(x^2+1)^2} - \left(-\frac{2x^2+1}{4(x^2+1)^2}\right) = \frac{1}{4}$, which is just a constant.

(Lecture 22) Integration by parts

$$\boxed{\int u\ dv = uv - \int v\ du}$$

Example:

$$\int \ln x\ dx = x\ln x - \int x\ d(\ln x) = x\ln x - \int x\cdot\frac{1}{x}\ dx = x\ln x - x + C$$

Example: $\int xe^x\ dx = \int x\ de^x = xe^x - \int e^x dx = xe^x - e^x + C$

More generally, for $\int x^n f(x)\ dx$:

- If $f(x) = \sin x, \cos x, e^x$ etc. (easy to integrate), try
$$\int x^n f(x)\ dx = \int x^n d(F(x)) = x^n F(x) - \int F(x) d(x^n)$$
- If $f(x) = \sin^{-1} x, \cos^{-1} x, \ln x$ etc. (hard to integrate) , try
$$\int x^n f(x)\ dx = \int f(x) d(\frac{x^{n+1}}{n+1}) = \frac{x^{n+1} f(x)}{n+1} - \int \frac{x^{n+1}}{n+1} d(f(x))$$

(Lecture 22) Integration by parts

Other common techniques:

- Integration by parts + solving equation

  Example: $\int e^x \cos x \; dx = ?$

  Solution: We have

$$I = \int e^x \cos x \; dx = \int e^x \; d(\sin x) = e^x \sin x - \int \sin x de^x$$
$$= e^x \sin x - \int e^x \sin x \; dx = e^x \sin x + \int e^x \; d\cos x$$
$$= e^x \sin x + e^x \cos x - \int \cos x \; de^x$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x \; dx$$

  Therefore, we have $I = e^x \sin x + e^x \cos x - I + C$ (as the two indefinite integrals may differ by a constant) and hence

$$I = \frac{e^x \sin x + e^x \cos x}{2} + C$$

(Lecture 22) Integration by parts

Other common techniques:

- Substitution + integration by parts

Example: $\int \cos(\ln x)\ dx = ?$

Solution: Let $u = \ln x$, then

$$du = \frac{1}{x} dx \Longrightarrow dx = x\ du = e^u du.$$

Therefore,

$$\begin{aligned}
\int \cos(\ln x) dx &= \int \cos u \cdot e^u du \\
&= \frac{e^u \sin u + e^u \cos u}{2} + C \\
&= \frac{x \sin(\ln x) + x \cos(\ln x)}{2} + C
\end{aligned}$$

(Lecture 22) Reduction formula

For integrals of the form

$$I_n = \int \cos^n x \; dx, \; \int \sin^n x \; dx, \int x^n \cos x \; dx, \; \int x^n \sin x \; dx,$$
$$\int x^n e^x \; dx, \int (\ln x)^n \; dx, \; \int e^x \cos^n x \; dx, \; \int e^x \sin^n x \; dx,$$
$$\int \frac{1}{(x^2+a^2)^n} \; dx, \; \int \frac{1}{(a^2-x^2)^n} \; dx \text{ etc.},$$

use integration by parts to write I_n in terms of some I_k with $k < n$.

Example:

$$I_n = \int x^n e^x \; dx = \int x^n \; d(e^x) = x^n e^x - \int e^x \; d(x^n)$$
$$= x^n e^x - \int n x^{n-1} e^x \; dx = x^n e^x - nI_{n-1}$$

So $\int x^{10} e^x \; dx = I_{10} = x^{10} e^x - 10 I_9 = x^{10} e^x - 10(x^9 e^x - 9 I_8) = \cdots$
(We can continue the process and eventually get some simple integral)

(Lecture 22) Partial fraction

Rational function: $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x), g(x)$ are polynomials

Examples: $\dfrac{x^4}{x^2+1}, \dfrac{2x+1}{3x^2+4x+1}, \dots$

Partial fraction decomposition:

Goal: Express $R(x) = q(x) + (\text{some simple fractions})$

1. Extract $q(x)$ first (if $\deg(f(x)) \geq \deg(g(x))$).
2. Factorize $g(x)$ into a product of linear polynomials (in the form of $ax+b$) and irreducible quadratic polynomials (in the form of ax^2+bx+c with $b^2-4ac<0$).
3. Write down the general terms in the partial fraction:

Factors of $g(x)$	Terms in partial fraction
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Bx+C}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \dots + \frac{B_kx+C_k}{(ax^2+bx+c)^k}$

4. Determine the coefficients A_i, B_i, C_i

(Lecture 22) Partial fraction

Example: Note that $\frac{9x-13}{x^2+x-12} = \frac{9x-13}{(x+4)(x-3)}$. Therefore, we have

$$\frac{9x-13}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3} = \frac{(A+B)x + (-3A+4B)}{(x+4)(x-3)}.$$

Comparing coefficients, $\begin{cases} A+B &= 9 \\ -3A+4B &= -13 \end{cases} \implies A = 7, B = 2.$

Therefore, $\frac{9x-13}{x^2+x-12} = \frac{7}{x+4} + \frac{2}{x-3}$.

Example: Note that $\frac{x^2+20x+11}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}$.

Therefore, we have

$$x^2 + 20x + 11 = A(x+1)(x-3) + B(x-3) + C(x+1)^2.$$

Putting $x = 3$, we get $C = 5$.
Putting $x = -1$, we get $B = 2$.
Putting $x = 0$, we get $11 = -3A + 2(-3) + 5(1)^2 \implies A = -4$.
Therefore, $\frac{x^2+20x+11}{(x+1)^2(x-3)} = -\frac{4}{x+1} + \frac{2}{(x+1)^2} + \frac{5}{x-3}$.

(Lecture 22) Partial fraction

Example:

$$\frac{4x^6+x^4-1}{x^4-1} = \frac{4x^6-4x^2+x^4-1+4x^2}{x^4-1}$$
$$= 4x^2+1+\frac{4x^2}{x^4-1} = 4x^2+1+\frac{4x^2}{(x-1)(x+1)(x^2+1)}$$

Therefore, we have

$$\frac{4x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$
$$= \frac{A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1)}{(x-1)(x+1)(x^2+1)}$$
$$= \frac{(A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+(A-B-D)}{(x-1)(x+1)(x^2+1)}$$

Comparing coefficients,

$$\begin{cases} A+B+C &= 0 \\ A-B+D &= 4 \\ A+B-C &= 0 \\ A-B-D &= 0 \end{cases} \implies A=1,\ B=-1,\ C=0,\ D=2.$$

Therefore, $\dfrac{4x^6+x^4-1}{x^4-1} = 4x^2+1+\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{2}{x^2+1}$.

(Lecture 22) Integration of partial fractions

Useful results for integrating partial fractions:

- $\int \frac{1}{ax+b}\, dx = \frac{1}{a} \ln|ax+b| + C$
- $\int \frac{1}{(ax+b)^k} dx$ (where $k > 1$) $= \frac{(ax+b)^{-k+1}}{a(-k+1)} + C$
- $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$
- $\int \frac{1}{(x^2+a^2)^k} dx$: use reduction formula/integration by parts
- $\int \frac{1}{ax^2+bx+c} dx$, $\int \frac{1}{(ax^2+bx+c)^k} dx$: write $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$, then use the above results
- $\int \frac{Ax+B}{ax^2+bx+c} dx =$ $\frac{A}{2a} \ln|ax^2+bx+c| + \left(B - \frac{Ab}{2a}\right) \int \frac{1}{ax^2+bx+c} dx$

(Lecture 22) Integration of rational functions

Example: $\displaystyle\int \frac{4x^6+x^4-1}{x^4-1}\,dx = ?$

Solution: By partial fraction decomposition, we have

$$\frac{4x^6+x^4-1}{x^4-1} = 4x^2+1+\frac{1}{x-1}-\frac{1}{x+1}+\frac{2}{x^2+1}$$

and hence

$$\begin{aligned}
&\int \frac{4x^6+x^4-1}{x^4-1}\,dx \\
&= \int \left(4x^2+1+\frac{1}{x-1}-\frac{1}{x+1}+\frac{2}{x^2+1}\right)\,dx \\
&= \frac{4x^3}{3}+x+\ln|x-1|-\ln|x+1|+2\tan^{-1}x+C
\end{aligned}$$

(Lecture 23) t-substitution

For $\int f(x)\,dx$ where $f(x)$ is a rational function in terms of $\cos x, \sin x, \tan x$, we can substitute $t = \tan\frac{x}{2}$. Then we have

$$\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \tan x = \frac{2t}{1-t^2},\ dx = \frac{2}{1+t^2}dt$$

and so $\int f(x)\,dx$ becomes an integral of rational function in t.

Example: $\displaystyle\int \frac{1}{2+\cos x}\,dx = ?$

Solution: Let $t = \tan\frac{x}{2}$. We have

$$\begin{aligned}
\int \frac{1}{2+\cos x}\,dx &= \int \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt \\
&= \int \frac{2}{2(1+t^2)+(1-t^2)}\,dt \\
&= \int \frac{2}{t^2+3}\,dt \\
&= 2\tan^{-1}\frac{t}{\sqrt{3}} + C = 2\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{3}} + C
\end{aligned}$$

(Lecture 23–24) Definite integration

Definite integral:

Let $a \leq b$ and $f(x)$ be a continuous function on $[a, b]$.

The definite integral $\int_a^b f(x)\, dx$ is defined as the signed area under the graph of $y = f(x)$ between $x = a$ and $x = b$.

Theorem (definite integral = limit of Riemann sum):

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $\Delta x_k = x_k - x_{k-1}$.

In particular, if all x_k are equally spaced, we have

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + \frac{k}{n}(b-a)\right) \cdot \frac{b-a}{n}$$

Example:

$$\int_0^1 x^2 dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$$

(Lecture 23–24) Definite integration

Properties of definite integrals:

- $\int_a^a f(x)\ dx = 0$
- $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$
- $\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
- $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx \quad \text{if } c \in (a, b)$
- For consistency, we also define $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

(Lecture 24) Fundamental theorem of calculus (FTC)

First fundamental theorem of calculus:

> Let $f(t)$ be a continuous function. Then
> $$\frac{d}{dx}\left(\int_a^x f(t)\ dt\right) = f(x).$$

Example: If $g(x) = \int_1^x \sin(t^2+5)\ dt$, find $g'(1)$.

Solution: By 1st FTC, $g'(x) = \sin(x^2+5) \Longrightarrow g'(1) = \sin 6$.

Example: If $g(x) = \int_0^{\sin x} e^{t^3}\ dt$, find $g'(x)$.

Solution: By 1st FTC,

$$g'(x) = \frac{d}{du}\left(\int_0^u e^{t^3}\ dt\right) \cdot \frac{du}{dx} \ (\text{let } u = \sin x)$$
$$= e^{u^3} \cdot \cos x = e^{\sin^3 x} \cos x.$$

(Lecture 24) Fundamental theorem of calculus (FTC)

Second fundamental theorem of calculus:

Suppose $F'(x) = f(x)$ on $[a, b]$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Example: $\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$

Definite integral by substitution:

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Example: $\int_0^1 \sqrt{2x+1}\,dx = ?$

Solution: Let $u = 2x + 1$, then $\frac{du}{dx} = 2 \Rightarrow du = 2\,dx$. Also, when $x = 0$ we have $u = 1$, and when $x = 1$ we have $u = 3$. Therefore,

$$\int_0^1 \sqrt{2x+1}\,dx = \int_1^3 \sqrt{u}\,\frac{1}{2}du = \left[\frac{u^{\frac{3}{2}}}{3}\right]_1^3 = \frac{3\sqrt{3}}{3} - \frac{1}{3} = \sqrt{3} - \frac{1}{3}$$

(Lecture 24) Fundamental theorem of calculus (FTC)

Integration by parts for definite integral:

$$\int_a^b uv' \, dx = [uv]_a^b - \int_a^b vu' \, dx$$

<u>Example:</u> $\int_1^2 \ln x \, dx = [x \ln x]_1^2 - \int_1^2 x(\ln x)' \, dx =$
$(2\ln 2 - 0) - \int_1^2 1 \, dx = 2\ln 2 - (2-1) = 2\ln 2 - 1$

Derivative of functions defined by definite integrals:

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t) \, dt\right) = f(v(x))v'(x) - f(u(x))u'(x)$$

<u>Example:</u> $\frac{d}{dx}\left(\int_{-\sin x}^{x^3} e^{t^2} \, dt\right) =$
$e^{(x^3)^2} \cdot 3x^2 - e^{(-\sin x)^2} \cdot (-\cos x) = 3x^2 e^{x^6} + e^{\sin^2 x} \cos x$

(Lecture 24) Evaluating limits by integrals

By treating the limit below as the limit of a Riemann sum, we have:

$$\boxed{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\ dx}$$

Example: $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right) = ?$

Solution: Note that $\displaystyle\frac{1}{\sqrt{n^2+k^2}} = \frac{1}{n}\cdot\frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}}$. We have

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}} = \int_0^1 \frac{1}{\sqrt{1+x^2}}\ dx\\
&= \left[\ln\left|\sqrt{1+x^2}+x\right|\right]_0^1 \text{ (using trigonometric substitution } x=\tan\theta)\\
&= \ln(\sqrt{2}+1)
\end{aligned}$$

(Lecture 24–25) Other definite integration techniques

- If f is an odd function, then $\int_{-a}^{a} f(x)\, dx = 0$.

  Example: $\int_{-2023}^{2023} x^4 \sin x \sin 2x \sin 3x \, dx = 0$

- If f is an even function, then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$.

  Example: $\int_{-1}^{1} x^2 \cos x \, dx = 2\int_{0}^{1} x^2 \cos x \, dx$

- Other symmetry arguments

  Example: $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx = ?$

  Solution: Let $u = \frac{\pi}{2} - x$, then $du = -dx$ and so $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx$

  $$= \int_{\frac{\pi}{2}}^{0} \frac{\sin^3\left(\frac{\pi}{2} - u\right)}{\sin^3\left(\frac{\pi}{2} - u\right) + \cos^3\left(\frac{\pi}{2} - u\right)} (-1)du = \int_0^{\frac{\pi}{2}} \frac{\cos^3 u}{\sin^3 u + \cos^3 u} du.$$

  Therefore,

  $$\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos^3 u}{\sin^3 u + \cos^3 u} du = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} dx$$

  $$= \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2} \implies \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx = \frac{\pi}{4}.$$

Good luck on your final exam!