MATH 1010F Lecture 24 28/11/2023

Last time

- useful results for integrating partial fractions
- t-substitution: $\boxed{t = \tan\frac{x}{2}}$
- Definite integral (Riemann integral)

  $\int_a^b f(x)\,dx =$ signed area under the graph of $y=f(x)$ between $x=a$ and $x=b$

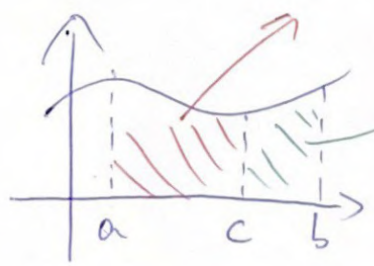
- Thm: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k$

  where $a = x_0 < x_1 < \cdots < x_n = b$

  and $\Delta x_k = x_k - x_{k-1}$.

Prop

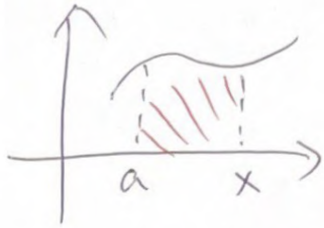
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$
- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ if $c\in(a,b)$.



Remark For consistency, we also define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

(where $a \le b$)

Fundamental theorem of Calculus (FTC)



Idea: Note that

$$\int_a^x f(t)\,dt = \text{signed Area under the graph of } f(t) \text{ on } a \leq t \leq x$$

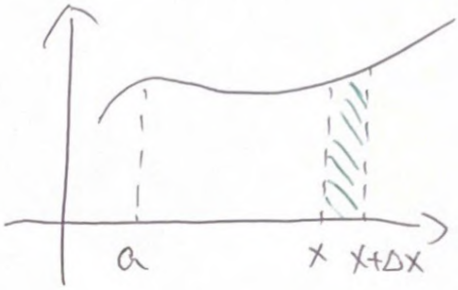
t ← dummy variable

$$= \underbrace{A(x)}$$

x ← actual variable that we are interested in.

↖ (area as a function: if x changes, the total area will also change)

Then we have



$$A(x+\Delta x) - A(x)$$

$$= \text{Area of }$$ 

$$\approx f(x)\cdot \Delta x$$

$$\Rightarrow \frac{A(x+\Delta x)-A(x)}{\Delta x} \approx f(x)$$

Taking $\Delta x \to 0$, we have: $\underbrace{\lim_{\Delta x \to 0} \frac{A(x+\Delta x)-A(x)}{\Delta x}} = f(x)$

↖ $= A'(x)$!

Thm (1st FTC)

Let $f(t)$ be a continuous function.

Then $\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$.

Example If $g(x) = \int_1^x \ln(t^2+1)\,dt$, find $g'(2)$.

Solution By 1^{st} FTC, $g'(x) = \ln(x^2+1)$

$$\therefore g'(2) = \ln(2^2+1) = \ln 5 \quad //$$

(Note: no need to explicitly calculate $\int_1^x \ln(t^2+1)\,dt$ first!)

Example If $g(x) = \int_1^{x^2} \cos(t^2)\,dt$, find $g'(x)$.

Solution Let $u = x^2$. We have

$$g'(x) = \frac{d}{du}\left(\int_1^u \cos(t^2)\,dt\right)\cdot\frac{du}{dx} \quad \text{(by Chain rule)}$$

$$= \cos(u^2)\cdot 2x \quad \text{(by } 1^{st} \text{ FTC)}$$

$$= \cos(x^4)\cdot 2x \quad //$$

Thm (2^{nd} FTC)

Suppose $F'(x) = f(x)$ on $[a,b]$. (i.e. $F(x)$ is an antiderivative of $f(x)$)

Then $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$.

Proof Let $g(x) = \int_a^x f(t)\,dt + F(a)$.

By 1^{st} FTC, $g'(x) = \left(\int_a^x f(t)\,dt + F(a)\right)'$

$$= f(x) + 0$$

$$= f(x) = F'(x)$$

$\therefore \quad g(x) = F(x) + C$, where C is a const.

Put $x = a$, $\quad g(a) = F(a) + C$

$$0 + F(a) = F(a) + C$$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad g(x) = F(x)$$

Now, $\quad F(b) = g(b)$

$$= \int_a^b f(t)\,dt + F(a)$$

$$\Rightarrow \quad F(b) - F(a) = \int_a^b f(t)\,dt$$

just dummy variables here

$$= \int_a^b f(x)\,dx \quad //$$

Remark From the 2^{nd} FTC,

finding area (definite integral) = finding antiderivative and taking their difference at endpoints

(related to indefinite integral and differentiation)

Example $\int_1^2 x^2\,dx$

$$= \left[\frac{x^3}{3}\right]_1^2 \quad \text{(finding antiderivative of } x^2\text{)}$$

$$= \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3} \quad //$$

Prop (Definite integral by substitution)

Let $u = u(x)$. We have

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

Example $\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = ?$

Solution Let $u = \sqrt{x}$, then $\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow du = \frac{1}{2\sqrt{x}}\, dx$

when $x = 0$, $u = 0$

when $x = \pi^2$, $u = \pi$

$$\therefore \int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = \int_0^{\pi^2} \sin\sqrt{x} \cdot 2 \cdot \frac{1}{2\sqrt{x}}\, dx$$

$$= \int_0^{\pi} 2\sin u\, du$$

$$= \left[-2\cos u\right]_0^{\pi}$$

$$= (-2\cos\pi) - (-2\cos 0)$$

$$= 2 + 2 = 4 \quad //$$

Prop (Integration by parts for definite integral)

$$\boxed{\int_a^b u v'\, dx = \left[uv\right]_a^b - \int_a^b v u'\, dx}$$

Example $\int_1^2 \ln x \, dx$

$$= \left[x \ln x\right]_1^2 - \int_1^2 x \, (\ln x)' \, dx$$

$$= (2\ln 2 - 1\ln 1) - \int_1^2 x \cdot \frac{1}{x} \, dx$$

$$= (2\ln 2 - 0) - \int_1^2 1 \, dx$$

$$= 2\ln 2 - \left[x\right]_1^2 = 2\ln 2 - (2-1)$$

$$= 2\ln 2 - 1 \quad //$$

Prop (Derivative of functions defined by definite integrals)

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\, dt = f(v(x))\, \underline{\underline{v'(x)}} - f(u(x))\, \underline{\underline{u'(x)}}$$

Proof By FTC, $\int_{u(x)}^{v(x)} f(t)\, dt = F(v(x)) - F(u(x))$

(where F is an antiderivative of f)

$$\therefore \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\, dt = \frac{d}{dx}\Big(F(v(x)) - F(u(x))\Big)$$

$$= F'(v(x)) \cdot v'(x) - F'(u(x)) \cdot u'(x)$$

(by chain rule)

$$= f(v(x))\, v'(x) - f(u(x))\, u'(x)$$

($\because F' = f$)

$//$

Example $\frac{d}{dx}\left(\int_0^{\sin x} \sqrt{1+t^4}\, dt\right)$

$$= \sqrt{1+(\sin x)^4} \cdot (\sin x)' - \sqrt{1+0^4} \cdot 0'$$

$$= \cos x \sqrt{1+\sin^4 x} \quad //$$

Example $\frac{d}{dx}\left(\int_{-x}^{x^2} e^{t^2} dt\right)$

$$= e^{(x^2)^2} \cdot (x^2)' - e^{(-x)^2} \cdot (-x)'$$

$$= 2x e^{x^4} + e^{x^2} \quad //$$

Evaluating limits by integrals

For limits in the form of $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(0+\frac{k}{n}(1-0)\right) \cdot \frac{1-0}{n}$$

$$= \int_0^1 f(x)\, dx$$

Example $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right) = ?$

Solution $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right)$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}$$

$$= \int_0^1 \frac{1}{1+x}\, dx = \left[\ln|1+x|\right]_0^1 = \ln 2 \quad //$$

Example $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\dots+\frac{n}{n^2+n^2}\right)=?$

Solution Note that $\frac{n}{n^2+k^2}=\frac{1}{n}\cdot\frac{n^2}{n^2+k^2}$

$$=\frac{1}{n}\cdot\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$\therefore \lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\dots+\frac{n}{n^2+n^2}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$=\int_0^1\frac{1}{1+x^2}\,dx$$

$$=\left[\tan^{-1}x\right]_0^1$$

$$=\tan^{-1}1-\tan^{-1}0=\frac{\pi}{4}$$ //

Example $\lim_{n\to\infty}\frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}=?$

Solution We have

$$\ln\left(\frac{\sqrt[n]{(n+1)\cdots(2n)}}{n}\right)=\frac{1}{n}\ln\left(\frac{(n+1)(n+2)\cdots(2n)}{n^n}\right)$$

$$=\frac{1}{n}\ln\left(\frac{n+1}{n}\cdot\frac{n+2}{n}\cdot\dots\cdot\frac{2n}{n}\right)$$

$$=\frac{1}{n}\ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)$$

$$=\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\dots+\ln\left(1+\frac{n}{n}\right)\right]$$

$$\therefore \ln\left(\lim_{n\to\infty}\frac{\sqrt[n]{(n+1)\cdots(2n)}}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\cdots+\ln\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)$$

$$= \int_0^1 \ln(1+x)\,dx$$

$$= \Big[x\ln(1+x)\Big]_0^1 - \int_0^1 x\,(\ln(1+x))'\,dx \quad \left(\begin{array}{c}\text{integration}\\ \text{by parts}\end{array}\right)$$

$$= (1\cdot\ln 2 - 0) - \int_0^1 \frac{x}{1+x}\,dx$$

$$= (\ln 2 - 0) - \int_0^1\left(1-\frac{1}{1+x}\right)dx$$

$$= \ln 2 - \Big[x - \ln|1+x|\Big]_0^1$$

$$= \ln 2 - \Big[(1-\ln 2)-(0-\ln 1)\Big] = 2\ln 2 - 1$$

$$\therefore \lim_{n\to\infty}\frac{\sqrt[n]{(n+1)\cdots(2n)}}{n} = e^{2\ln 2 - 1} = \frac{4}{e} \quad //$$

Other definite integration techniques

- If f is an <u>odd</u> function (i.e. $f(-x) = -f(x)$ for all x), then $\int_{-a}^{a} f(x)\,dx = 0$

<u>Proof</u> $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_{-a}^{0} f(x)\,dx$

$$= \int_0^a f(x)\,dx + \int_a^0 f(-u)\cdot(-1)\,du \quad \left(\begin{array}{c}\text{sub.}\\ u=-x\end{array}\right)$$

$$= \int_0^a f(x)\,dx + \int_a^0 f(u)\,du \quad (\because -f(-u)=f(u))$$

$$= \int_0^a f(x)\,dx - \int_0^a f(u)\,du = 0 \quad //$$

- If f is even (i.e. $f(x)=f(-x)$ for all x),

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

Example $\displaystyle\int_{-5}^5 \frac{x^3\sin^2 x}{x^4+2x^2+1}\,dx = ?$

Solution Note that for $f(x) = \dfrac{x^3\sin^2 x}{x^4+2x^2+1}$,

$$f(-x) = \frac{(-x)^3\sin^2(-x)}{(-x)^4+2(-x)^2+1} = -\frac{x^3\sin^2 x}{x^4+2x^2+1} = -f(x)$$

$\therefore f$ is odd.

$\therefore \displaystyle\int_{-5}^5 f(x)\,dx = 0 \quad //$