MATH 1010 F Lecture 23 — 23/11/2023

Last time

- Integration by parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

- Reduction formula
- Integration of rational functions
  - partial fraction decomposition:

$$\frac{f(x)}{g(x)} = q(x) + (\text{some simple fractions})$$

polynomial

in the form of

$$\frac{A_1}{ax+b}, \ldots, \frac{A_k}{(ax+b)^k}, \quad \frac{B_1x+C_1}{ax^2+bx+c}, \ldots, \frac{B_kx+C_k}{(ax^2+bx+c)^k}$$

  - Then

$$\int \frac{f(x)}{g(x)}\,dx = \int q(x)\,dx + \int (\text{partial fractions})\,dx$$

Useful results for integrating partial fractions:

- $$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$
- $$\int \frac{1}{(ax+b)^k}\,dx \quad (\text{where } k>1) \quad = \frac{(ax+b)^{-k+1}}{a(-k+1)} + C$$

- $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

- $\displaystyle\int \frac{1}{(x^2+a^2)^k}\,dx$: use reduction formula and/or Integration by parts

- $\displaystyle\int \frac{1}{ax^2+bx+c}\,dx$, $\displaystyle\int \frac{1}{(ax^2+bx+c)^k}\,dx$: similar to above

(note: $\displaystyle\frac{1}{ax^2+bx+c} = \frac{1}{a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}} = \frac{1}{a}\cdot\frac{1}{u^2+d^2}$

where $u = x+\frac{b}{2a}$

$d = \frac{4ac-b^2}{a^2}$)

- $$\int \frac{Ax+B}{ax^2+bx+c}\,dx = \int \frac{\frac{A}{2a}(2ax+b)+B-\frac{Ab}{2a}}{ax^2+bx+c}\,dx$$
$$= \frac{A}{2a}\ln\left|ax^2+bx+c\right| + \left(B-\frac{Ab}{2a}\right)\int\frac{1}{ax^2+bx+c}\,dx$$

use above

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t - substitution

Idea: To find $\int f(x)\,dx$ where $f(x)$ is a rational function in terms of $\cos x$, $\sin x$, $\tan x$, we can substitute $\boxed{t = \tan\frac{x}{2}}$

Then $\tan x = \frac{2t}{1-t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, and $dx = \frac{2}{1+t^2}\,dt$, and $\int f(x)\,dx$ becomes an integral of rational function in t.

Example $\int \frac{1}{\sin x}\,dx$

$$= \int \frac{1}{\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{1}{t}\,dt = \ln|t| + C = \ln\left|\tan\frac{x}{2}\right| + C \quad //$$

Example $\int \frac{1}{1-\cos x}\,dx$

$$= \int \frac{1}{1-\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1+t^2-(1-t^2)}\,dt$$

$$= \int \frac{1}{t^2}\,dt = -\frac{1}{t} + C = -\cot\frac{x}{2} + C \quad //$$

Example $\int \frac{1}{1+\cos x+\sin x}\,dx$

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{1}{1+t}\,dt$$

$$= \ln|1+t| + C = \ln\left|1+\tan\frac{x}{2}\right| + C \quad //$$

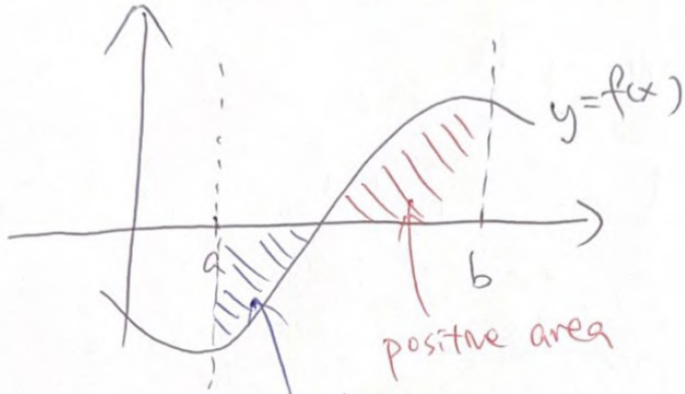
Remark May also need to combine with partial fraction decomposition.

Definite integral

Def (definite integral)

Let $a \leq b$ and $f(x)$ be a continuous function on $[a,b]$.
We define the definite integral

$$\boxed{\int_a^b f(x)\,dx = \text{signed area under the graph of } y=f(x) \text{ between } x=a \text{ and } x=b.}$$



$\int_a^b f(x)\,dx$

= Area of 

− Area of 

Remark
- For now, we consider $\int_a^b f(x)\,dx$ as something totally irrelevant to the indefinite integral $\int f(x)\,dx$!
- We will show their connection later

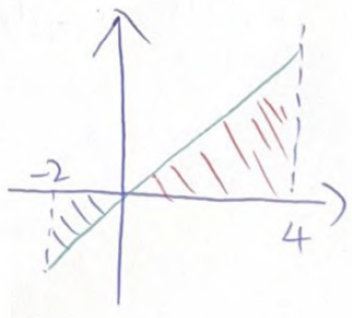
Example $f(x) = x$

$\int_{-2}^{4} f(x)\,dx$ = Area of 

− Area of 

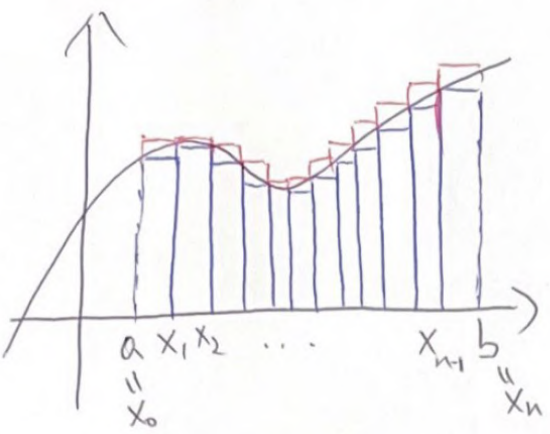
$= \frac{1}{2}\cdot 4^2 - \frac{1}{2}\cdot 2^2$

$= 8 - 2 = 6$ //



How to find the area for more complicated functions?

Consider <u>upper sum</u> and <u>lower sum</u> of rectangles :



upper sum: approximate from above, may overestimate the area

lower sum: approximate from below, may underestimate the area

We have

$$\underbrace{\text{lower sum}} \le \int_a^b f(x)\,dx \ (\underbrace{\text{actual area}}) \le \underbrace{\text{upper sum}}$$

By considering a finer and finer partition (i.e. $n \to \infty$, with the bin width $\to 0$), we have :

<u>Thm</u> $$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k}$$

where $a = x_0 < x_1 < x_2 < \dots < x_n = b$

and $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \dots, n$.

In particular, if all x_k are equally spaced,

we have $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\Big(\underbrace{a + \tfrac{k}{n}(b-a)}_{\text{the point } x_k}\Big)\underbrace{\left(\frac{b-a}{n}\right)}_{\text{spacing}}$$

<u>Example</u> $\displaystyle\int_{0 \leftarrow a}^{1 \leftarrow b} x^2 \, dx$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\Big(\overset{a}{0} + \frac{k}{n}(\overset{b}{1} - \overset{a}{0})\Big)\Big(\frac{\overset{b}{1} - \overset{a}{0}}{n}\Big)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \cdot \left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} = \frac{1\cdot 2}{6} = \frac{1}{3} \quad //$$

<u>Next time</u> : <u>Fundamental thm of Calculus</u>

(linking the concept of indefinite integral /antiderivative
with the concept of definite integral)