MATH 1010F Lecture 22 21/11/2023

Last time

- Trigonometric integrals
  - use trigonometric identities
  - $\int \cos^m x \sin^n x \, dx$
  - $\int \sec^m x \tan^n x \, dx$
- Trigonometric substitution
  - For $\sqrt{a^2-x^2}$, sub. $x = a\sin\theta$
  - For $\sqrt{a^2+x^2}$, sub. $x = a\tan\theta$
  - For $\sqrt{x^2-a^2}$, sub. $x = a\sec\theta$

Example $\int \frac{1}{\sqrt{1+x^2}} dx = ?$

Solution Let $x = \tan\theta$.

Then we have $dx = \sec^2\theta \, d\theta$

and $\sec^2\theta = 1+\tan^2\theta = 1+x^2$

$$\therefore \int \frac{1}{\sqrt{1+x^2}} dx = \int \frac{\sec^2\theta \, d\theta}{\sqrt{\sec^2\theta}}$$

$$= \int \sec\theta \, d\theta$$

$$= \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta} d\theta$$

$$= \int \frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta} d\theta$$

$$= \int \frac{1}{\sec\theta + \tan\theta}\, d(\sec\theta + \tan\theta) \quad \left(\because (\sec\theta + \tan\theta)' = \sec\theta\tan\theta + \sec^2\theta\right)$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{1+x^2} + x\right| + C \quad //$$

<u>Example</u> $\int \sqrt{1-x^2}\, dx = ?$

<u>Solution</u> Let $x = \sin\theta$.

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2\theta}\, d(\sin\theta)$$

$$= \int \cos\theta \cdot \cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\, d\theta \quad \left(\because \cos 2\theta = 2\cos^2\theta - 1\right)$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

$$= \frac{\sin^{-1}x}{2} + \frac{2\sin\theta\cos\theta}{4} + C$$

$$= \frac{\sin^{-1}x}{2} + \frac{x\sqrt{1-x^2}}{2} + C \quad //$$

<u>Integration by parts</u>

<u>Thm</u> (Integration by parts)

Let $u(x)$, $v(x)$ be differentiable functions.

We have

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Proof Note that

$$\frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

$$\therefore \quad u \cdot \frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx} \cdot v$$

$$\therefore \quad \int u \frac{dv}{dx}\, dx = \int \frac{d}{dx}(uv)\, dx - \int \frac{du}{dx} \cdot v\, dx$$

$$\Rightarrow \int u\, dv = uv - \int v\, du \quad //$$

Example $\int x e^x dx$

$$= \int x\, de^x \qquad (\because (e^x)' = e^x)$$

$$= \underset{u}{\underset{\uparrow}{x}}\,\underset{v}{\underset{\uparrow}{e^x}} - \int \underset{v}{\underset{\uparrow}{e^x}}\, d\underset{u}{\underset{\uparrow}{x}} \qquad \text{(integration by parts)}$$

$$= x e^x - e^x + C \quad //$$

Example $\int \ln x\, dx$

$$= x \ln x - \int x\, d(\ln x) \qquad \text{(integration by parts)}$$

$$= x \ln x - \int x \cdot \frac{1}{x}\, dx \qquad (\because (\ln x)' = \frac{1}{x})$$

$$= x \ln x - \int 1\, dx$$

$$= x \ln x - x + C \quad //$$

Example $\int x \sin x \, dx$

$$= \int x \, d(-\cos x)$$

$$= -\int x \, d(\cos x)$$

$$= -\left(x\cos x - \int \cos x \, dx \right)$$

$$= -(x\cos x - \sin x) + C$$

$$= \sin x - x\cos x + C \quad //$$

Remark The choice of "u" and "v" will affect the result!

e.g. $\int x \sin x \, dx$

$$= \int \sin x \, d\left(\frac{x^2}{2}\right)$$

$$= \frac{1}{2} \int \sin x \, d(x^2)$$

$$= \frac{1}{2}\left(x^2 \sin x - \int x^2 \, d(\sin x) \right)$$

$$= \frac{1}{2}\left(x^2 \sin x - \int x^2 \cos x \, dx \right) = \cdots$$

getting more complicated!

More generally, for $\boxed{\int x^n f(x) \, dx}$:

① If $f(x) = \sin x$, $\cos x$, e^x, try $\int x^n \, d(F(x))$

then we have $x^n F(x) - \int F(x) \, d(x^n) = x^n F(x) - \int n x^{n-1} F(x) dx$

lower power!

② If $f(x) = \sin^{-1} x$, $\cos^{-1} x$, $\ln x$, try

$$\int f(x) \, d\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\left(x^{n+1} f(x) - \int x^{n+1} \, d(f(x)) \right) = \cdots$$

<u>Example</u> $\int x^n \ln x \, dx$

$$= \int \ln x \, d\left(\frac{x^{n+1}}{n+1}\right)$$

$$= \frac{1}{n+1}\left(x^{n+1}\ln x - \int x^{n+1} d(\ln x)\right) \quad \text{(integration by parts)}$$

$$= \frac{1}{n+1}\left(x^{n+1}\ln x - \int x^{n+1} \cdot \frac{1}{x} \, dx\right)$$

$$= \frac{1}{n+1}\left(x^{n+1}\ln x - \int x^n \, dx\right)$$

$$= \frac{1}{n+1}\left(x^{n+1}\ln x - \frac{x^{n+1}}{n+1}\right) + C$$

$$= \frac{x^{n+1}}{(n+1)^2}\left((n+1)\ln x - 1\right) + C \quad //$$

<u>Example</u> $\int x^2 \sin x \, dx$

$$= \int x^2 \, d(-\cos x)$$

$$= -\left(x^2 \cos x - \int \cos x \, d(x^2)\right) \quad \text{(integration by parts)}$$

$$= -\left(x^2 \cos x - \int 2x \cos x \, dx\right)$$

$$= -\left(x^2 \cos x - 2\int x \, d(\sin x)\right)$$

$$= -\left(x^2 \cos x - 2\left(x \sin x - \int \sin x \, dx\right)\right) \quad \text{(integration by parts once more)}$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C \quad //$$

Other common techniques:

- **Integration by parts + solving equation:**

Example $I = \int e^x \cos x \, dx$

$$= \int e^x \, d(\sin x)$$

$$= e^x \sin x - \int \sin x \, d(e^x) \qquad \text{(integration by parts)}$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x \sin x + \int e^x \, d(\cos x)$$

$$= e^x \sin x + e^x \cos x - \int \cos x \, d(e^x) \qquad \text{(integration by parts)}$$

$$= e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x \, dx} \quad \leftarrow \text{same as the original one!}$$

$$\therefore I = e^x \sin x + e^x \cos x - I + C$$

$$\Rightarrow I = \frac{1}{2}(e^x \sin x + e^x \cos x) + C \quad \leftarrow \text{const.}$$

- **Substitution + integration by parts:**

Example $\int \cos(\ln x) \, dx = ?$

Sol Let $u = \ln x$, then $du = \frac{1}{x} dx$

$$\Rightarrow dx = x \, du = e^u du$$

$$\therefore \int \cos(\ln x) \, dx = \int \cos u \cdot e^u du$$

$$= \frac{e^u \sin u + e^u \cos u}{2} + C \quad \text{(by last example)}$$

$$= \frac{x \sin(\ln x) + x \cos(\ln x)}{2} + C$$

Reduction formula

Idea: For integral of the forms

$$I_n = \int \cos^n x\,dx,\ \int \sin^n x\,dx,\ \int x^n \cos x\,dx,\ \int x^n \sin x\,dx,$$

$$\int x^n e^x\,dx,\ \int (\ln x)^n\,dx,\ \int e^x \cos^n x\,dx,\ \int e^x \sin^n x\,dx,$$

$$\int \frac{1}{(x^2+a^2)^n}\,dx,\ \int \frac{1}{(a^2-x^2)^n}\,dx \quad \text{etc.},$$

use integration by parts to write I_n in terms of some I_k with $k<n$.

Example $I_n = \int x^n e^x\,dx$

$$\begin{aligned}
&= \int x^n\, d(e^x) \\
&= x^n e^x - \int e^x\, d(x^n) \\
&= x^n e^x - \int n x^{n-1} e^x\,dx \\
&= x^n e^x - n I_{n-1}
\end{aligned}$$

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Example $I_n = \int \cos^n x\,dx$

$$\begin{aligned}
&= \int \cos^{n-1} x\, d(\sin x) \\
&= \cos^{n-1} x \sin x - \int \sin x\, d(\cos^{n-1} x) \\
&= \cos^{n-1} x \sin x - \int (\sin x)(n-1)\cos^{n-2} x \cdot (-\sin x)\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx
\end{aligned}$$

$$= \cos^{n-1}x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$\therefore I_n = \cos^{n-1}x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow n I_n = \cos^{n-1}x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n} I_{n-2} \quad //$$

$$= \frac{\cos^{n-3}x \sin x}{n-2} + \frac{n-3}{n-2} I_{n-4}$$

$$= \cdots$$

Integration of rational functions

Def (Rational functions)

A rational function is a function of the form

$$\boxed{R(x) = \frac{f(x)}{g(x)}}$$

where $f(x)$, $g(x)$ are polynomials with real coefficients and $g(x) \neq 0$

Example $\frac{x}{x^2+3x+1}$, $\frac{x^3-x+1}{x^2+1}$, ...

Thm Let $R(x) = \frac{f(x)}{g(x)}$ be a rational function.

① (Division algorithm for polynomials)

There exists polynomials $q(x), r(x)$ with $\deg(r(x)) < \deg(g(x))$ s.t.

$$\underbrace{R(x) = q(x)}_{\text{quotient}} + \frac{r(x)}{g(x)} \leftarrow \text{remainder}$$

② (Fundamental theorem of algebra)

$g(x)$ can be factorized as a product of linear and/or irreducible quadratic polynomials.

(i.e. ax^2+bx+c with $\Delta = b^2-4ac<0$)

Partial fractions

Idea: Motivated by the above thm, we want to express $R(x) = q(x) + $ (some simple fractions)

Steps: ① Factorize $g(x)$.

② Write down the general terms:

Factor of $g(x)$	Terms in partial fractions
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$(ax^2+bx+c)^k$	$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx+C_k}{(ax^2+bx+c)^k}$

③ Determine the coefficients A_i, B_i, C_i

Example $\dfrac{9x-13}{x^2+x-12}$

Solution

① $x^2+x-12 = (x+4)(x-3)$

② Let $\dfrac{9x-13}{x^2+x-12} = \dfrac{A}{x+4} + \dfrac{B}{x-3}$

③ $\dfrac{9x-13}{x^2+x-12} = \dfrac{A(x-3)+B(x+4)}{(x+4)(x-3)} = \dfrac{(A+B)x-3A+4B}{(x+4)(x-13)}$

$$\therefore \begin{cases} A+B=9 \\ -3A+4B=-13 \end{cases}$$

$$\Rightarrow A=7,\ B=2$$

$$\therefore \frac{9x-13}{x^2+x-12} = \frac{7}{x+4} + \frac{2}{x-3}$$

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Example $\dfrac{x^2+20x+11}{(x+1)^2(x-3)}$

Solution Let $\dfrac{x^2+20x+11}{(x+1)^2(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-3}$

$\Rightarrow x^2+20x+11 = A(x+1)(x-3) + B(x-3) + C(x+1)^2$

Can substitute any number to find the coefficients, as the equation should hold for any x.

Put $x=3 \Rightarrow 80 = 16C \Rightarrow C=5$

Put $x=-1 \Rightarrow -8=-4B \Rightarrow B=2$

Put $x=0 \Rightarrow 11 = -3A-3B+C = -3A-1 \Rightarrow A=-4$

$$\therefore \frac{x^2+20x+11}{(x+1)^2(x-3)} = \frac{-4}{x+1} + \frac{2}{(x+1)^2} + \frac{5}{x-3}$$

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Example $\dfrac{4x^2+14x-9}{(x^2+x+1)(x-2)}$

Solution Note: For x^2+x+1, $\Delta = b^2-4ac = 1^2-4(1)(1) = -3 < 0$

$\therefore$ irreducible

Let $\dfrac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \dfrac{Ax+B}{x^2+x+1} + \dfrac{C}{x-2}$

$$\Rightarrow 4x^2+14x-9 = (Ax+B)(x-2) + C(x^2+x+1)$$

$$= (A+C)x^2 + (-2A+B+C)x + (C-2B)$$

$$\therefore \begin{cases} A+C = 4 \\ -2A+B+C = 14 \\ C-2B = -9 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 7 \\ C = 5 \end{cases}$$

$$\therefore \frac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \frac{-x+7}{x^2+x+1} + \frac{5}{x-2} \quad //$$

Remark If $R(x) = \dfrac{f(x)}{g(x)}$ with $\deg(f(x)) \geq \deg(g(x))$,

extract the higher order term first:

e.g. $\dfrac{2x^3}{x^2-1} = \dfrac{2x^3-2x+2x}{x^2-1}$

$$= \frac{2x(x^2-1)+2x}{x^2-1}$$

$$= 2x + \frac{2x}{x^2-1}$$

$$= 2x + \frac{2x}{(x+1)(x-1)} = 2x + \frac{1}{x-1} + \frac{1}{x+1}$$

With the partial fraction decomposition method above, we have

$$\int \frac{f(x)}{g(x)}\,dx = \underbrace{\int (\text{polynomial})\,dx}_{\text{very easy}} + \underbrace{\int \begin{pmatrix}\text{partial}\\ \text{fractions}\end{pmatrix} dx}_{\text{recall standard results}}$$

Example $\dfrac{4x^6+x^4-1}{x^4-1} = 4x^2+1+\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{2}{x^2+1}$

$$\therefore \int \frac{4x^6+x^4-1}{x^4-1}\,dx = \int \left((4x^2+1)+\frac{1}{x-1}-\frac{1}{x+1}+\frac{2}{x^2+1}\right)dx$$

$$= \frac{4x^3}{3}+x+\ln|x-1|-\ln|x+1| + 2\tan^{-1}x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$