MATH 1010F Lecture 21 16/11/2023

Last time
- Taylor series
  - Differentiation
  - Finding limits
- Integration
  - Indefinite integral: $\int f(x)\,dx = F(x) + C$
  - Basic properties $\left(\int (f \pm g)\,dx,\ \int (kf)\,dx\right)$
  - Integration by substitution

Trigonometric integrals

Useful identities:

- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$
- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
- $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
- $\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$
- $\cos x \sin y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$
- $\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

Example

$$\int \sin^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)dx \quad \left(\because \cos 2x = 1 - 2\sin^2 x\right)$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C \quad //$$

<u>Example</u> $\int \cos^4 x \, dx$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \quad (\because \cos 2x = 2\cos^2 x - 1)$$

$$= \int \frac{1+2\cos 2x + \cos^2 2x}{4} dx$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \int \frac{1+\cos 4x}{8} dx \quad \left(\begin{aligned} \because\ &\cos 4x \\ &= \cos(2(2x)) \\ &= 2\cos^2 2x \\ &\quad -1 \end{aligned}\right)$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C \quad //$$

<u>Example</u> $\int \sin 5x \cos 3x \, dx$

$$= \int \frac{1}{2}\left(\sin \overset{5x+3x}{\overset{\downarrow}{8x}} + \sin \overset{5x-2x}{\overset{\downarrow}{2x}}\right) dx$$

$$= -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C \quad //$$

<u>Example</u> $\int \cos x \cos^2 3x \, dx$

$$= \int \cos x \left(\frac{1}{2}(1+\cos 6x)\right) dx$$

$$= \frac{1}{2} \int (\cos x + \cos x \cos 6x) dx$$

$$= \frac{1}{2}\sin x + \frac{1}{2}\int \left(\frac{1}{2}\left(\cos \overset{6x+x}{\overset{\downarrow}{7x}} + \cos \overset{6x-x}{\overset{\downarrow}{5x}}\right)\right) dx$$

$$= \frac{1}{2}\sin x + \frac{1}{28}\sin 7x + \frac{1}{20}\sin 5x + C$$

//

Technique for finding $\boxed{\int \cos^m x \sin^n x \, dx}$

(where m, n are nonnegative integers):

① If m is odd, sub $u = \sin x$.

Then we have $du = \cos x \, dx$

and $\cos^2 x = 1 - \sin^2 x = 1 - u^2$

Example $\int \cos^3 x \sin^3 x \, dx$

$$= \int \underbrace{\cos^2 x}\, \underbrace{\sin^3 x}\, \underbrace{\cos x \, dx}$$

$$= \int \underbrace{(1-u^2)}\, \underbrace{u^3}\, \underbrace{du}$$

$$= \int (u^3 - u^5)\, du$$

$$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$ //

② If n is odd, sub $u = \cos x$

Then we have $du = -\sin x \, dx$

and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$

Example $\int \sin^5 x \, dx$

$$= \int \sin^4 x \sin x \, dx$$

$$= \int (1-u^2)^2 \cdot (-du)$$

$$= -\int (1 - 2u^2 + u^4)\, du$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$ //

③ If both m, n are even, use double angle formulas to reduce the power:

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos x \sin x = \frac{\sin 2x}{2}.$$

Then use ① and/or ② if applicable.

<u>Example</u> $\int \sin^4 x \cos^2 x \, dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \cdot \frac{1+\cos 2x}{2} \, dx$$

$$= \frac{1}{8} \int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{1}{8}\int \frac{1+\cos 4x}{2} \, dx + \frac{1}{8} \underbrace{\int \cos^3 2x \, dx}_{\text{use ①}}$$

$$= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{x}{16} - \frac{\sin 4x}{64} + \underbrace{\frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)}_{} + C$$

$$= \frac{1}{8}x - \frac{1}{48}\sin^3 2x - \frac{1}{64}\sin 4x + C \quad //$$

Similarly, for $\boxed{\int \sec^m x \tan^n x \, dx}$:

① If m is even, sub $u = \tan x$

Then $du = \sec^2 x \, dx$ and $\sec^2 x = 1 + \tan^2 x = 1 + u^2$

$$\Rightarrow \int \sec^m x \tan^n x \, dx$$

$$= \int \sec^{2k} x \tan^n x \, dx$$

$$= \int \sec^{2k-2} x \tan^n x \, d(\tan x)$$

$$= \int (1+u^2)^{k-1} u^n \, du$$

② If n is odd, sub $u = \sec x$

then $du = \sec x \tan x \, dx$ and $\tan^2 x = \sec^2 x - 1$
$= u^2 - 1$

$$\Rightarrow \int \sec^m x \tan^n x \, dx$$
$$= \int \sec^m x \tan^{2k+1} x \, dx$$
$$= \int \sec^{m-1} x \tan^{2k} x \, d(\sec x)$$
$$= \int u^{m-1} (u^2-1)^k \, du$$

③ If m is odd and n is even, use $\tan^2 x = \sec^2 x - 1$ to write everything in terms of $\sec x$.

Trigonometric substitution

Idea : • For $\sqrt{a^2-x^2}$, sub $x = a \sin\theta$

• For $\sqrt{a^2+x^2}$, sub $x = a \tan\theta$

• For $\sqrt{x^2-a^2}$, sub $x = a \sec\theta$

(here $a>0$)

Example $\int \frac{dx}{\sqrt{9-x^2}} = ?$

Solution Let $x = 3\sin\theta$, then $dx = 3\cos\theta \, d\theta$

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta \, d\theta}{\sqrt{9-9\sin^2\theta}} = \int \frac{3\cos\theta \, d\theta}{3\cos\theta}$$
$$= \int d\theta$$
$$= \theta + C = \sin^{-1}\frac{x}{3} + C \;//$$