MATH1010F University Mathematics

Review: Differentiation and Applications of Differentiation

Gary Choi

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https://www.math.cuhk.edu.hk/course/2324/math1010f

Quiz 2 reminder

- Date: November 16 (this Thursday)
- Time: 5:35PM - 6:20PM
- Venue for MATH1010F: YIA LT2
- Closed book, closed notes
- Bring student ID card, black/blue pen
- List of approved calculators:
  `http://www.res.cuhk.edu.hk/images/content/examinations/use-of-calculators-during-course-examination/Use-of-Calculators-during-Course-Examinations.pdf`

Scope (see Blackboard announcement)

1. **Differentiation**:
   - Differentiability of functions
   - Derivatives of exponential, logarithmic, and trigonometric functions
   - Differentiation rules (sum, difference, product, quotient, chain rules)
   - Derivatives of piecewise-defined functions, continuous but not differentiable functions, functions with discontinuous derivatives
   - Implicit differentiation, logarithmic differentiation
   - Derivatives of inverse functions
   - Higher order derivatives
2. **Applications of differentiation**:
   - Extremum values of functions
   - Increasing and decreasing functions, proving inequalities
   - Concavity, points of inflection, asymptotes (horizontal, vertical, oblique)
   - Curve sketching
   - Rolle's theorem, Lagrange's mean value theorem, Cauchy's MVT
   - L'Hopital's rule
   - Taylor polynomials and Taylor series

(Lecture 8) Differentiability of functions

f is said to be differentiable at $x = a$ if the following limit (called the derivative of f at $x = a$) exists:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Another form:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Remark: For piecewise functions, we need to check both

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \text{ and } \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

Example of finding derivative by definition (i.e. **first principle**):

- If $f(x) = x^2$, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x. \end{aligned}$$

(Lecture 8–9) Derivatives of polynomial, exponential, logarithmic, and trigonometric functions

- $(x^n)' = nx^{n-1}$
- $(e^x)' = e^x$
- $(\ln x)' = \dfrac{1}{x}$
- $(a^x)' = a^x \ln a$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x = \dfrac{1}{\cos^2 x}$
- $(c)' = 0$ (where c is a constant)
- $(\sinh x)' = \cosh x$ (where $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$)
- $(\cosh x)' = \sinh x$
- $(\tanh x)' = \text{sech}^2 x = \dfrac{1}{\cosh^2 x}$

(Lecture 8–9) Differentiation rules (sum, difference, product, and quotient rules)

If f and g are differentiable at a point, then the following functions are also differentiable at that point:

- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(cf(x))' = cf'(x)$ (where c is a constant)
- Product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

- Quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad (\text{if } g(x) \neq 0)$$

Examples:

- $(x^3 \sin x)' = (x^3)' \sin x + x^3(\sin x)' = 3x^2 \sin x + x^3 \cos x$
- $\left(\frac{\sin x}{x^2+1}\right)' = \frac{(\sin x)'(x^2+1)+(\sin x)(x^2+1)'}{(x^2+1)^2} = \frac{(x^2+1)\cos x+2x\sin x}{(x^2+1)^2}$

(Lecture 8–9) Differentiation rules (chain rule)

Chain rule:

If $f(x)$ is differentiable at $x = a$ and $g(u)$ is differentiable at $u = f(a)$, then $(g \circ f)$ is differentiable at $x = a$ and we have

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

In other words, we have

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Examples:

- $(\sin x^2)' = \frac{d(\sin u)}{du}\frac{du}{dx}$ (let $u = x^2$) $= (\cos u)(2x) = 2x \cos x^2$
- $(e^{\sin x})' = e^{\sin x} \cos x$

A more complicated version: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

Example:

- $\left(\ln(\cos(x^3))\right)' = \dfrac{1}{\cos x^3} \cdot (-\sin(x^3)) \cdot (3x^2) = -3x^2 \tan x^3$

(Lecture 8–9) Continuity and differentiability

Property:

If f is differentiable at $x = a$, then f is continuous at $x = a$

The converse is **NOT** true: if f is continuous at $x = a$, it may or may not be differentiable at $x = a$

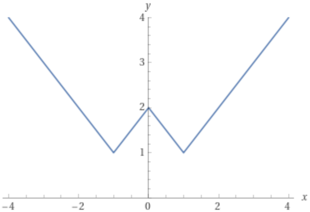
Example: $f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

- $f(x)$ is continuous on $\mathbb{R}$ (i.e. at every point $x \in \mathbb{R}$):
  - For any $a < 0$, $\lim_{x \to a} f(x) = \lim_{x \to a} (-x) = -a = f(a)$
  - For any $a > 0$, $\lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a)$
  - For $a = 0$, we have $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0 = f(0)$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 = f(0)$, and hence $\lim_{x \to 0} f(x) = f(0)$
- $f(x)$ is not differentiable at $x = 0$:
  Note that $f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h \to 0} \dfrac{|h| - 0}{h} = \lim_{h \to 0} \dfrac{|h|}{h}$ but
  $\lim_{h \to 0^+} \dfrac{|h|}{h} = \lim_{h \to 0^+} \dfrac{h}{h} = 1$ and $\lim_{h \to 0^-} \dfrac{|h|}{h} = \lim_{h \to 0^+} \dfrac{-h}{h} = -1$

(Lecture 8–9) Continuity and differentiability

Another example of continuous but not differentiable functions:

$$f(x) = |x+1| - |x| + |x-1|$$
$$= \begin{cases} -(x+1) - (-x) - (x-1) & = -x & \text{if } x < -1 \\ (x+1) - (-x) - (x-1) & = x+2 & \text{if } -1 \le x < 0 \\ (x+1) - (x) - (x-1) & = -x+2 & \text{if } 0 \le x < 1 \\ (x+1) - (x) + (x-1) & = x & \text{if } x \ge 1 \end{cases}$$



- $f(x)$ is continuous on $\mathbb{R}$
- $f(x)$ is not differentiable at $x = -1, 0, 1$

(Lecture 10–11) Implicit differentiation

Idea: Find y' without having to explicitly write $y = f(x)$.

Example:
If $x \sin y + y^2 = x + 3y$, find the slope of tangent at $(0, 0)$.

$$\begin{aligned}
(x \sin y + y^2)' &= (x + 3y)' \\
(\sin y + x(\cos y)y') + 2yy' &= 1 + 3y' \\
(x \cos y + 2y - 3)y' &= 1 - \sin y \\
y' &= \frac{1 - \sin y}{x \cos y + 2y - 3}
\end{aligned}$$

The slope of tangent at $(0, 0)$ is $\dfrac{1 - \sin 0}{0 \cdot \cos 0 + 2 \cdot 0 - 3} = -\dfrac{1}{3}$

(Lecture 10–11) Logarithmic differentiation

Idea: Find the derivative of some complicated functions using logarithms.

Example: If $y = x^x$, find y'.

$$\begin{aligned}
y &= x^x \\
\ln y &= \ln(x^x) \\
\ln y &= x \ln x \\
(\ln y)' &= (x \ln x)' \\
\frac{1}{y} y' &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\
y' &= y(\ln x + 1) = x^x(\ln x + 1)
\end{aligned}$$

(Lecture 10–11) Derivatives of inverse functions

Inverse functions:
If $f(y)$ is a bijective and differentiable function with $f'(y) \neq 0$ for any y, then the inverse function $y = f^{-1}(x)$ is differentiable:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Examples:

$$y = \sin^{-1} x \Rightarrow \sin y = x \Rightarrow (\cos y)y' = 1 \Rightarrow (\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$y = \cos^{-1} x \Rightarrow \cos y = x \Rightarrow (-\sin y)y' = 1 \Rightarrow (\cos^{-1} x)' = -\frac{1}{\sqrt{1 - x^2}}$$

$$y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow (\sec^2 y)y' = 1 \Rightarrow (\tan^{-1} x)' = \frac{1}{1 + x^2}$$

(Lecture 11–12) Higher order derivatives

- **Second derivative**:

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

- *n*-**th derivative**:

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\frac{dy}{dx}\right)\right)\right)$$

- 0-**th derivative**:

$$y^{(0)} = f^{(0)}(x) = f(x)$$

Examples:

- $(\sin x^2)'' = ((\sin x^2)')' = ((\cos x^2)(2x))'$
  $= (-\sin x^2)(2x)(2x) + 2\cos x^2 = -4x^2 \sin x^2 + 2\cos x^2$
- Find y'' if $xy + \sin y = 1$:

$$(xy + \sin y)' = 1' \Rightarrow (y + xy' + y'\cos y) = 0 \Rightarrow y' = \frac{-y}{x + \cos y}$$

$$\Rightarrow y'' = -\frac{y'(x + \cos y) - y(1 - y'\sin y)}{(x + \cos y)^2} = \frac{2y(x + \cos y) + y^2 \sin y}{(x + \cos y)^3}$$

(Lecture 11–12) Higher order differentiation rules

If f and g are n-times differentiable (i.e. $f^{(n)}$ and $g^{(n)}$ exist), then:

- $(f \pm g)^{(n)} = f^{(n)} \pm g^{(n)}$
- $(cf)^{(n)} = cf^{(n)}$ (where c is a constant)
- Leibniz's rule (product rule for higher order derivatives):

$$\boxed{(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}}$$

where $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$ is the binomial coefficient.

Example: $(x^3 \sin x)^{(4)}$

$= 1 \cdot (x^3)'''' \sin x + 4 \cdot (x^3)'''(\sin x)' + 6 \cdot (x^3)''(\sin x)'' + 4(x^3)'(\sin x)'''$
$\quad +1 \cdot x^3 (\sin x)''''$

$= 0 + 24 \cos x - 36x \sin x - 12x^2 \cos x + x^3 \sin x$

$= (x^3 - 36x) \sin x + (24 - 12x^2) \cos x$

(Lecture 12–13) n-times differentiability and continuity

If f is n-times differentiable at $x = a$
($f^{(n)}(a)$ exists, i.e. $f^{(n-1)}$ is differentiable at $x = a$),
then $f^{(n-1)}$ is continuous at $x = a$.

f is n-times differentiable at $x = a$ (i.e. $f^{(n)}(a)$ exists)

$$\Downarrow$$

$f^{(n-1)}(a)$ exists and $f^{(n-1)}$ is continuous at $x = a$

$$\Downarrow$$

$$\vdots$$

$$\Downarrow$$

$f'(a)$ exists and f' is continuous at $x = a$

$$\Downarrow$$

f is continuous at $x = a$

However, the converse is **NOT** true!

Example: Let $f(x) = |x|x$, then:

- f is differentiable at $x = 0$
- f' is continuous at $x = 0$
- but f' is not differentiable at $x = 0$ (i.e. $f''(0)$ does not exist)

(Lecture 14) Local extrema, critical points, turning points

Local maximum:

$f(x)$ has a local maximum at $x = a$ if $f(x) \leq f(a)$ for all x near a (more precisely, for all $x \in D \cap (a - \delta, a + \delta)$ where D is the domain and $\delta > 0$ is some small number).

Local minimum:

$f(x)$ has a local minimum at $x = a$ if $f(x) \geq f(a)$ for all x near a.

Note:

Local extremum points can be either interior points or endpoints!

Example: For $f : [-\pi, \pi] \to \mathbb{R}$ with $f(x) = \sin x$,

local maximum points = $(-\pi, 0)$, $(\frac{\pi}{2}, 1)$

local minimum points = $(-\frac{\pi}{2}, -1)$, $(\pi, 0)$.

Critical points:

f has a critical point at $x = a$ if $f'(a) = 0$ or $f'(a)$ does not exist.

Turning points:

f has a turning point at $x = a$ if f' changes sign at a.

Note: $\{\text{Turning points}\} \subset \{\text{Critical points}\}$

Example: $x = 0$ is a critical point of $f(x) = x^3$, but it is not a turning point.

(Lecture 14) First and second derivative tests

Theorem:

Let $f(x)$ be a continuous function. If $f(x)$ has a local maximum/ minimum at $x = a$, then $x = a$ must be a critical point of $f(x)$.

First derivative test:

Let $f(x)$ be a continuous function and $x = a$ be a critical point.

(i) If f' changes sign from $+$ to $-$ at a, then $f(x)$ has a local maximum at $x = a$.

(ii) If f' changes sign from $-$ to $+$ at a,then $f(x)$ has a local minimum at $x = a$.

Second derivative test:

Let $f(x)$ be a continuous function.

(i) If $f'(a) = 0$ and $f''(a) < 0$, then $f(x)$ has a local maximum at $x = a$.

(ii) If $f'(a) = 0$ and $f''(a) > 0$, then $f(x)$ has a local minimum at $x = a$.

(Lecture 15) Finding global extrema

Extreme value theorem (EVT) for closed and bounded intervals:

> Let f be a continuous function on $[a, b]$. Then there exists $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$ (i.e. f has a global maximum and a global minimum in $[a, b]$).

Note: For f on (a, b), $(a, b]$, or $[a, b)$, f may NOT have any global extrema in some cases!

Finding global extrema for functions on general intervals:

1. Check all critical points (including endpoints if applicable) to find all local extrema.
2. Compare the values of $f(x)$ at all such points as well as the limit of f as x approaches the open endpoints (if applicable) to determine the existence of global extrema.

Examples:

$f(x) = x^2$ on $[-2, 1]$: global min. point $= (0, 0)$; global max. $= (-2, 4)$
$f(x) = x^2$ on $\mathbb{R}$: global minimum point $= (0, 0)$; no global max.
$f(x) = x^2$ on $(0, 1)$: no global min; no global max

(Lecture 15) Concavity and points of inflection

Concavity:
We say that $f(x)$ is

- concave upward on (a, b) if $f''(x) > 0$ on (a, b)
- concave downward on (a, b) if $f''(x) < 0$ on (a, b)

Example: $f(x) = x^3 \implies f''(x) = 6x$
f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$

Point of inflection:
We say that $x = a$ is an inflection point of $f(x)$ if $f''(x)$ changes sign at $x = a$.
Example: $f(x) = x^3 \implies f''(x) = 6x$
As f'' changes sign from $-$ to $+$ at $x = 0$, f has an inflection point at $x = 0$.

(Lecture 15) Asymptotes (vertical, horizontal, oblique)

Vertical asymptotes:

- $x = a$ is a vertical asymptote of $f(x)$ if

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

Example: For $f(x) = x^2 + \dfrac{1}{x-1}$,
$x = 1$ is a vertical asymptote since $\lim_{x \to 1^+} f(x) = \infty$.

Horizontal asymptotes:

- $y = b$ is a horizontal asymptote of $f(x)$ if

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b$$

Note: $f(x)$ can have at most two different horizontal asymptotes (one for $\lim_{x \to -\infty}$ and one for $\lim_{x \to \infty}$)

Example: For $f(x) = e^x$,
$y = 0$ is a horizontal asymptote since $\lim_{x \to -\infty} f(x) = 0$.

Oblique asymptotes:

- $y = ax + b$ is an oblique asymptote of $f(x)$ if

$$\lim_{x\to-\infty}(f(x) - (ax+b)) = 0 \text{ or } \lim_{x\to\infty}(f(x) - (ax+b)) = 0$$

- Note: $f(x)$ can have at most two different oblique asymptotes (one for $\lim\limits_{x\to-\infty}$ and one for $\lim\limits_{x\to\infty}$)

Example: For $f(x) = x + 3 + \dfrac{2}{x}$, $y = x + 3$ is an oblique asymptote since $\lim\limits_{x\to\infty}(f(x) - (x+3)) = \lim\limits_{x\to\infty} \dfrac{2}{x} = 0.$

- Finding oblique asymptotes:
  Method 1: Directly work on $f(x) - (ax + b)$, then check the coefficients of different terms and see what a, b have to be such that the limit $= 0$ as $x \to \infty$ or $-\infty$.
  Method 2: Find a such that $a = \lim\limits_{x\to\infty} \dfrac{f(x)}{x}$ (or $\lim\limits_{x\to-\infty}$), then find $b = \lim\limits_{x\to\infty}(f(x) - ax)$ (or $\lim\limits_{x\to-\infty}$).

(Lecture 15) Asymptotes (vertical, horizontal, oblique)

Example: $f(x) = \sqrt{x^2 - 2x + 3}$

- No vertical asymptote (as $f(x)$ is defined everywhere on $\mathbb{R}$)
- No horizontal asymptote ($\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$)
- Oblique asymptotes:

For $x \to \infty$, we have

$$a = \lim_{x\to\infty} \frac{\sqrt{x^2 - 2x + 3}}{x} = \lim_{x\to\infty} \sqrt{1 - \frac{2}{x} + \frac{3}{x^2}} = 1, \text{ and}$$

$$b = \lim_{x\to\infty} (\sqrt{x^2 - 2x + 3} - x) = \lim_{x\to\infty} \frac{(x^2 - 2x + 3) - x^2}{\sqrt{x^2 - 2x + 3} + x} = -1$$

For $x \to -\infty$, we have

$$a = \lim_{x\to-\infty} \frac{\sqrt{x^2 - 2x + 3}}{x} = \lim_{x\to-\infty} -\sqrt{1 + \frac{2}{x} + \frac{3}{x^2}} = -1, \text{ and}$$

$$b = \lim_{x\to-\infty} (\sqrt{x^2 - 2x + 3} + x) = \lim_{x\to-\infty} \frac{(x^2 - 2x + 3) - x^2}{\sqrt{x^2 - 2x + 3} - x} = 1$$

So the oblique asymptotes are $y = x - 1$ and $y = -x + 1$.

(Lecture 15) Curve sketching

To sketch a given function, do the following:

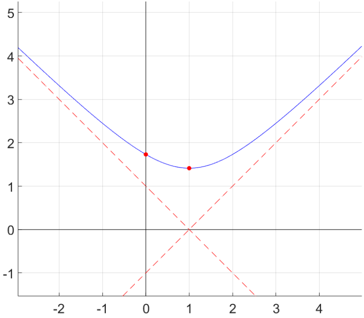
1. Find:
   - (Natural) domain
   - x-intercept
   - y-intercept
   - Asymptotes (vertical, horizontal, oblique)
   - Critical points (and check whether they are local max/min)
   - Inflection points (and check concavity)
2. Sketch the curve based on the information above.

Examples: See the main MATH1010 lecture notes.

(Lecture 15) Curve sketching

Example: $f(x) = \sqrt{x^2 - 2x + 3}$

- Domain: $\mathbb{R}$ (as $\sqrt{x^2 - 2x + 3} = \sqrt{(x-1)^2 + 2}$ is defined everywhere)
- x-intercept: None (as $f(x) = \sqrt{(x-1)^2 + 2} \neq 0$)
- y-intercept: $f(0) = \sqrt{3}$
- Asymptotes: $y = x - 1$ and $y = -x + 1$ (see the previous slide)
- Critical points: $f'(x) = \frac{x-1}{\sqrt{x^2-2x+3}}$, so the only critical point is at $x = 1$.
  By first derivative test, it is a local minimum.
- Inflection point: None (as $f''(x) = \frac{2}{\sqrt{x^2-2x+3}} > 0$)



(Lecture 15–17) Mean value theorem (MVT)

Rolle's theorem:

If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Lagrange's mean value theorem:

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Cauchy's mean value theorem:

If f, g are continuous on $[a, b]$, differentiable on (a, b), with $g(a) \neq g(b)$ and $g'(x) \neq 0$ on (a, b), then there exists $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Using MVTs to prove inequalities:

Example: Prove that $|\cos(x) - \cos(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Solution:

- If $x = y$, we have $|\cos(x) - \cos(y)| = 0 = |x - y|$.
- If $x \neq y$, by Lagrange's MVT, there exists c between x and y such that
$$\frac{\cos(x) - \cos(y)}{x - y} = -\sin(c).$$
Therefore, we have
$$\frac{|\cos(x) - \cos(y)|}{|x - y|} = |-\sin(c)| \leq 1 \iff |\cos(x) - \cos(y)| \leq |x - y|$$
for all $x, y \in \mathbb{R}$.

(Lecture 16) Derivatives and inequalities

Increasing/decreasing functions and derivatives:

- f is (monotonic) increasing on (a, b) (i.e. $f(x) \leq f(y)$ for all $x, y \in (a, b)$ with $x < y$) if and only if $f'(x) \geq 0$ on (a, b).
- f is (monotonic) decreasing on (a, b) (i.e. $f(x) \geq f(y)$ for all $x, y \in (a, b)$ with $x < y$) if and only if $f'(x) \leq 0$ on (a, b).
- f is constant on (a, b) if and only if $f'(x) = 0$ on (a, b).
- f is strictly increasing on (a, b) (i.e. $f(x) < f(y)$ for all $x, y \in (a, b)$ with $x < y$) **if** $f'(x) > 0$ on (a, b).
- f is strictly decreasing on (a, b) (i.e. $f(x) > f(y)$ for all $x, y \in (a, b)$ with $x < y$) **if** $f'(x) < 0$ on (a, b).

Using derivatives to prove inequalities:

Example: Let $p > 1$. Prove that $(1 + x)^p > 1 + px$ for all $x > 0$.

Solution: Let $f(x) = (1 + x)^p - (1 + px)$. Then

$$f'(x) = p(1 + x)^{p-1} - p > 0$$

for all $x > 0$. Therefore, f is strictly increasing on $(0, \infty)$. We have

$$f(x) > f(0) = 0 \Longrightarrow (1 + x)^p > 1 + px.$$

(Lecture 17) L'Hopital's rule

L'Hopital's rule:

Let $a \in \mathbb{R}$ or $a = \pm\infty$. If f and g are differentiable near a and all of the following conditions are satisfied:

1. Both $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ or both $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$.
2. $g'(x) \neq 0$ near a.
3. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or $= \pm\infty$.

Then we have $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Remarks:

- Similar results hold for one-sided limit ($\lim_{x \to a^-}$ and $\lim_{x \to a^+}$)
- Sometimes may need to apply the rule more than once
- Not always applicable! Check if the requirements are satisfied.

(Lecture 17) L'Hopital's rule

Handling different indeterminate forms:

- $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$: May try to apply the L'Hopital's rule directly

  Example:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} \ (\frac{0}{0}) = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \ (\frac{0}{0})$$

$$= \lim_{x\to 0} \frac{2\sec x \sec x \tan x}{6x} = \lim_{x\to 0} \frac{\sin x}{3x\cos^3 x} = \frac{1}{3}$$

- $0 \cdot (\pm\infty)$, $\infty - \infty$: May try to convert them into $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then apply the L'Hopital's rule

  Example:

$$\lim_{x\to 1}(x^2 - 1)\tan\frac{\pi x}{2} \ \ (0\cdot\infty) = \lim_{x\to 1} \frac{x^2 - 1}{\cot\frac{\pi x}{2}} \ \ (\frac{0}{0})$$

$$= \lim_{x\to 1} \frac{2x}{\frac{\pi}{2}\cdot\csc^2\frac{\pi x}{2}} = \lim_{x\to 1} \frac{2x\sin^2\frac{\pi x}{2}}{\frac{\pi}{2}} = \frac{2\cdot 1\cdot 1^2}{\frac{\pi}{2}} = \frac{4}{\pi}$$

(Lecture 17) L'Hopital's rule

Handling different indeterminate forms:

- 1^∞, ∞^0, 0^0: May use logarithm and apply the L'Hopital's rule to the logged expression, then use $\lim_{x\to a} y = e^{\lim_{x\to a} \ln y}$

  Example: Find $\lim_{x\to 0^+} (x+\sin x)^x \quad (0^0)$

  Solution: Let $y = (x+\sin x)^x$, then $\ln y = x\ln(x+\sin x)$ and

$$\begin{aligned}\lim_{x\to 0^+} x\ln(x+\sin x) \quad (0\cdot(\pm\infty)) &= \lim_{x\to 0^+} \frac{\ln(x+\sin x)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty}\right)\\ &= \lim_{x\to 0^+} \frac{\frac{1}{x+\sin x}(1+\cos x)}{-\frac{1}{x^2}}\\ &= \lim_{x\to 0^+} \frac{-x(1+\cos x)}{1+\frac{\sin x}{x}}\\ &= \frac{-0(1+1)}{1+1} = 0\end{aligned}$$

  So $\lim_{x\to 0^+} (x+\sin x)^x = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$

(Lecture 18) Taylor polynomial

Taylor polynomial:

The *n*-th order Taylor polynomial of $f(x)$ about a point $x = a$ is

$$p_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Property: We have $f^{(k)}(a) = p_n^{(k)}(a)$ for all $k = 0, 1, \ldots, n$.

Example:
The 2nd order Taylor polynomial of $f(x) = \sqrt{1+x}$ about $x = 0$ is

$$p_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 = 1 + \frac{x}{2} - \frac{x^2}{8}$$

Taylor's theorem:

Let $x \neq a$ (i.e. $x > a$ or $x < a$).
Suppose $f^{(n)}$ exists and is continuous on $[a, x]$ (or $[x, a]$), and $f^{(n+1)}$ exists on (a, x) (or (x, a)).
Then there exists $c \in (a, x)$ (or (x, a)) such that

$$f(x) = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

(Lecture 19–20) Taylor series

Taylor series:

The Taylor series of $f(x)$ about a point $x = a$ is the infinite series

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Property: If the remainder term in Taylor's theorem $R_n(x) \to 0$ as $n \to \infty$ on an interval I, then the Taylor series is equal to the function (i.e. $f(x) = T(x)$) on I.

Examples: $e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all $x \in \mathbb{R}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ for all $x \in \mathbb{R}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ for all $x \in \mathbb{R}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$ for $|x| < 1$

(Lecture 19–20) Taylor series

Properties:

- If $T(x)$ is the Taylor series of $f(x)$ about $x = 0$, then $T(x^k)$ is the Taylor series of $f(x^k)$ about $x = 0$ for all positive integer k

  Example: The Taylor series of $\frac{\sin x^2}{x^2}$ about 0 is

  $$\frac{1}{x^2}\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right) = 1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \cdots$$

- Addition and subtraction of Taylor series

  Example: The Taylor series of $\frac{\sin x^2}{x^2} + \cos x$ about 0 is

  $$\left(1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \cdots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 2 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

- Multiplication and division of Taylor series

  Example: The Taylor series of $\frac{\sin x^2}{x^2}\cos^3 x$ about 0 is

  $$\left(1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^3 = 1 - \frac{3x^2}{2} + \frac{17x^4}{24} + \cdots$$

(Lecture 19–20) Taylor series

Properties:

- Composition of Taylor series
  Example:
  The Taylor series of $\cos(\sin x)$ about 0 is
  $$1 - \frac{1}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \frac{1}{4!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^4 - \cdots$$
  $$= 1 - \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$$

- Differentiation of Taylor series
  Example:
  The Taylor series of $-\frac{x}{(1+x)^2} = x\left(\frac{1}{1+x}\right)'$ is
  $$x\left(1 - x + x^2 - x^3 - \cdots\right)'$$
  $$= x(-1 + 2x - 3x^2 + \cdots)$$
  $$= -x + 2x^2 - 3x^3 + \cdots$$

(Lecture 20) Using Taylor series to find limits

Idea: To find $\lim_{x \to c} f(x)$, replace certain components in $f(x)$ with their Taylor series (if those components are equal to their Taylor series for x near c)

Example:

$$\lim_{x\to 0} \frac{\ln(1+x) - x\sqrt{1-x}}{x - \sin x}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \mathcal{O}(x^4)\right) - x\left(1 - \frac{x}{2} - \frac{x^2}{8} + \mathcal{O}(x^3)\right)}{x - \left(x - \frac{x^3}{6} + \mathcal{O}(x^5)\right)}$$

$$= \lim_{x\to 0} \frac{\frac{11}{24}x^3 + \mathcal{O}(x^4)}{\frac{1}{6}x^3 + \mathcal{O}(x^5)}$$

$$= \lim_{x\to 0} \frac{\frac{11}{24} + \mathcal{O}(x)}{\frac{1}{6} + \mathcal{O}(x^2)} = \frac{\frac{11}{24} + 0}{\frac{1}{6} + 0} = \frac{11}{4}$$

Good luck!