MATH1010F Lecture 20

14/11/2023

<u>Last time</u>

- <u>Taylor's theorem</u>:

$$f(x) = \underbrace{p_n(x)}_{n^{\text{th}}\text{ order Taylor polynomial about } x=a} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{error term}}, \quad c \in (a,x)$$

- <u>Taylor series</u>:

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- If $\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \to 0$ as $n \to \infty$ on an interval I,

then $f(x) = T(x)$ on I.

- Properties: $+, -, \times, \div$, composition

Prop (Differentiation of Taylor series)

If the Taylor series of $f(x)$ about c is

$$T(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots,$$

then the Taylor series of $f'(x)$ about c is

$$\boxed{T'(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots}$$

Example $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$(\sin x)' = 1 - 3\cdot\frac{x^2}{3!} + 5\frac{x^4}{5!} - \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \cos x$$

Example Find the Taylor series of $\tan^{-1} x$ about 0.

Solution Let the Taylor series be

$$T(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

then

$$T'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

Now, note that $(\tan^{-1} x)' = \frac{1}{1+x^2}$

and the Taylor series of $\frac{1}{1+x^2}$ about 0 is

$$1 - x^2 + x^4 - x^6 + \dots$$

Comparing coefficients, $\begin{cases} a_1 = 1 & \\ 2a_2 = 0 & \Rightarrow a_2 = 0 \\ 3a_3 = -1 & \Rightarrow a_3 = -\frac{1}{3} \\ \vdots & \end{cases}$

$$\therefore T(x) = a_0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Finally, since $a_0 = \tan^{-1} 0 = 0$, we have

$$T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

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Remark Similar result holds for integration.

later !

Using Taylor series for computing limits

Idea: To find $\lim_{x \to c} f(x)$, we can replace certain components in $f(x)$ with their Taylor series about c if they are equal (at least in a small neighborhood of c).

Example $\lim_{x \to 0} \dfrac{e^x - 1 - \sin x}{\cos x - 1} = ?$

Solution Using Taylor series about 0, we have:

$$e^x - 1 - \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \frac{x^2}{2} + O(x^3)$$

higher order terms of degree ≥ 3

OK to have "=" here as $e^x = 1 + x + \frac{x^2}{2!} + \cdots$, $\sin x = x - \frac{x^3}{3!} + \cdots$, $\cos x = 1 - \frac{x^2}{2!} + \cdots$ for all $x \in \mathbb{R}$.

and
$$\cos x - 1 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - 1$$

$$= -\frac{x^2}{2} + O(x^4)$$

$$\therefore \frac{e^x - 1 - \sin x}{\cos x - 1} = \frac{\frac{x^2}{2} + O(x^3)}{-\frac{x^2}{2} + O(x^4)}$$

$$= \frac{\frac{1}{2} + O(x)}{-\frac{1}{2} + O(x^2)} \to \frac{\frac{1}{2} + 0}{-\frac{1}{2} + 0} = -1 \text{ as } x \to 0.$$

$$\therefore \lim_{x \to 0} \frac{e^x - 1 - \sin x}{\cos x - 1} = -1$$

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Integration

Idea:

$$\begin{tikzcd} F(x) \arrow[r, bend left, "\substack{\text{differentiation}\\ F'(x)}"] & f(x) \arrow[l, bend left] \end{tikzcd}$$

how to reverse the process?

Def (Indefinite integral)

Let $f(x)$ be a continuous function.

An anti-derivative of $f(x)$ (also known as "primitive function")

is a function $F(x)$ s.t. $F'(x) = f(x)$.

The collection of all anti-derivatives of $f(x)$ is called the indefinite integral of $f(x)$ and is denoted by

$$\int \underbrace{f(x)}_{\text{integrand}} \, dx .$$

integration sign

integration variable is x.

Remark Anti-derivatives are not unique!

$$\int f(x)\, dx = F(x) + C$$

any constant

Example

- $(x^3)' = 3x^2 \quad \therefore \int 3x^2\, dx = x^3 + C$
- $(\sin x)' = \cos x \quad \therefore \int \cos x\, dx = \sin x + C$

where C is a constant

Some basic integrals

- $\int k\,dx = kx + C$
- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
- $\int e^x\,dx = e^x + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \csc x \cot x\,dx = -\csc x + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int a^x\,dx = \frac{1}{\ln a} a^x + C$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$

also $= -\sin^{-1} x + C$

(note that $\cos^{-1} x = -\sin^{-1} x + \frac{\pi}{2}$, differ by a constant)

- $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

Prop For any continuous functions $f(x)$, $g(x)$ and any constant k,

- $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- $\int k f(x)\,dx = k \int f(x)\,dx$.

Example

- $\int (x^3 - x + 5)\,dx = \int x^3\,dx - \int x\,dx + \int 5\,dx$
  $= \frac{x^4}{4} - \frac{x^2}{2} + 5x + C$ //

- $\int \frac{(x+1)^2}{x}\,dx = \int \frac{x^2+2x+1}{x}\,dx$
  $= \int (x + 2 + \frac{1}{x})\,dx = \frac{x^2}{2} + 2x + \ln|x| + C$ //

- $\int (4x^2 - \csc^2 x - \frac{1}{1+x^2})\,dx$

$$= \frac{4x^3}{3} + \cot x - \tan^{-1} x + C \quad //$$

Example (Finding C explicitly)

Suppose $f'(x) = x^3 - 1$ and $f(2) = 1$. Find $f(x)$.

Solution $f'(x) = x^3 - 1 \Rightarrow f(x) = \int (x^3-1)\,dx = \frac{1}{4}x^4 - x + C$

$$f(2) = 1 \Rightarrow \frac{1}{4}(2)^4 - 2 + C = 1$$

$$4 - 2 + C = 1 \qquad \therefore f(x) = \frac{1}{4}x^4 - x - 1$$

$$C = -1 \quad //$$

Prop (Integration by substitution)

Let $f(u)$ be a function of u and

$u = u(x)$ be a function of x.

Then $$\boxed{\underbrace{\int f(u(x))\frac{du}{dx}\,dx}_{\text{in terms of } x} = \underbrace{\int f(u)\,du}_{\text{in terms of } u}}$$

Proof Let $F(u)$ be an anti-derivative of $f(u)$.

Then $\int f(u)\,du = F(u) + C$

By Chain rule, $(F(u(x)))' = F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x)$

$\therefore F(u(x))$ is an anti-derivative of $f(u(x)) \cdot u'(x)$

$$\therefore \int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du \quad //$$

Example $\int \sqrt{3x+4}\, dx = ?$

Solution Let $u = 3x+4$

then $\frac{du}{dx} = 3 \Rightarrow du = 3\,dx$

$$\int \sqrt{3x+4}\, dx = \frac{1}{3}\int \sqrt{3x+4} \cdot 3\,dx$$
$$= \frac{1}{3}\int \sqrt{u}\, du$$
$$= \frac{1}{3}\cdot\frac{2}{3} u^{\frac{3}{2}} + C$$
$$= \frac{2}{9}(3x+4)^{\frac{3}{2}} + C$$

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Example $\int e^{2x^2+1} x\, dx = ?$

Let $u = 2x^2+1$, then $\frac{du}{dx} = 4x \Rightarrow du = 4x\,dx$

$$\therefore \int e^{2x^2+1} x\, dx = \frac{1}{4}\int e^{2x^2+1} \cdot 4x\, dx$$
$$= \frac{1}{4}\int e^{u}\, du$$
$$= \frac{1}{4} e^{u} + C$$
$$= \frac{1}{4} e^{2x^2+1} + C$$

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Example $\int \frac{dx}{2x+1} = ?$

Solution Let $u = 2x+1$, then $du = 2\,dx$

$$\int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{2dx}{2x+1} = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln|2x+1| + C$$

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Example $\int \frac{(1+\ln x)^6}{x} dx = ?$

Solution Let $u = 1 + \ln x$, then $\frac{du}{dx} = \frac{1}{x}$

$$du = \frac{1}{x} dx$$

$$\therefore \int \frac{(1+\ln x)^6}{x} dx = \int (u)^6 du$$

$$= \frac{u^7}{7} + C$$

$$= \frac{(1+\ln x)^7}{7} + C \quad //$$

Example $\int \cos^4 x \sin x \, dx$

$$= \int \cos^4 x \, d(-\cos x) \leftarrow \left(\because \frac{d(-\cos x)}{dx} = \sin x\right)$$

$$= -\int \cos^4 x \, d(\cos x)$$

$$= -\frac{\cos^5 x}{5} + C$$

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