MATH1010F Lecture 19 7/11/2023

Last time

- n^{th} order Taylor polynomial of $f(x)$ about $x=a$

$$\boxed{\begin{aligned} p_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}}$$

- $p_n^{(k)}(a) = f^{(k)}(a)$ for all $k=0,1,\dots,n$.
- Larger $n \Rightarrow$ better approximation of f for x close enough to a
- May not be accurate at x far away from a

Thm (Taylor's theorem)

Let $x \neq a$ (i.e. $x > a$ or $x < a$).

Suppose $f^{(n)}$ exists and is continuous on $[a,x]$ (or $[x,a]$), and $f^{(n+1)}$ exists on (a,x) (or (x,a)).

Then there exists $c \in (a,x)$ (or (x,a)) s.t.

$$f(x) = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

remainder in Lagrange form

$$= f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Proof (Idea: use Rolle's thm repeatedly.)

Assume $x > a$ (proof for the case $x < a$ is similar).

① Let

$$F(y) = f(y) - p_n(y) - \frac{f(x) - p_n(x)}{(x-a)^{n+1}} (y-a)^{n+1}$$

here we consider y as the variable, with $y \in [a, x]$.

Check:

- $F(a) = f(a) - p_n(a) - 0 = 0$ (by prop. of $p_n(x)$)
- $F(x) = f(x) - p_n(x) - (f(x) - p_n(x)) = 0$

$\therefore$ By Rolle's thm, there exists $c_1 \in (a, x)$ s.t.

$$F'(c_1) = 0$$

derivative with respect to y.

② Consider

$$F'(y) = f'(y) - p_n'(y) - (n+1) \frac{f(x) - p_n(x)}{(x-a)^{n+1}} (y-a)^n$$

Check:

- $F'(a) = f'(a) - p_n'(a) - 0 = 0$ (by prop of $p_n(x)$)
- $F'(\underline{c_1}) = 0$ (by ①)

$\therefore$ We can apply Rolle's thm to F' on $(a, \underline{c_1})$.

to get $\underline{c_2} \in (a, \underline{c_1})$ s.t. $F''(\underline{c_2}) = 0$

③ Repeat the above process for

$$F''', F^{(4)}, \dots, F^{(n)}$$

$\Rightarrow$ there exists $c_{n+1} \in (a, c_n)$ s.t. $F^{(n+1)}(c_{n+1}) = 0$

$$\Rightarrow f^{(n+1)}(c_{n+1}) - 0 - (n+1)! \frac{f(x)-p_n(x)}{(x-a)^{n+1}} = 0$$

$$\Rightarrow \quad f(x) = p_n(x) + \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(x-a)^{n+1}$$

note: $c_{n+1} \in (a, c_n)$

$\Rightarrow c_{n+1} \in (a, x)$

<u>Example</u> $f(x) = \frac{1}{1-x}$

① Find the n^{th} order Taylor polynomial of f about 0.

② For what value of n, we can ensure that the approximation error of $p_n(x)$ at $x=0.1$ is less than 10^{-10}, i.e. $|f(0.1) - p_n(0.1)| < 10^{-10}$?

<u>Solution</u> ①

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$\vdots$$

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$$

$$\therefore f^{(k)}(0) = k!$$

$$\therefore p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = 1 + x + x^2 + \dots + x^n.$$

② By Taylor's theorem,

$$f(0.1) = p_n(0.1) + \frac{f^{(n+1)}(c)}{(n+1)!}(0.1-0)^{n+1}$$

for some $c \in (0, 0.1)$.

$$\therefore \left| f(0.1) - P_n(0.1) \right| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (0.1-0)^{n+1} \right|$$

$$= \left| \frac{\frac{(n+1)!}{(1-c)^{n+2}}}{(n+1)!} (0.1)^{n+1} \right|$$

$$= \frac{(0.1)^{n+1}}{(1-c)^{n+2}}$$

$$< \frac{(0.1)^{n+1}}{0.9^{n+2}} \quad (\because 0 < c < 0.1)$$

$$= 10 \cdot \left(\frac{1}{9}\right)^{n+2}$$

$$10 \cdot \left(\frac{1}{9}\right)^{n+2} < 10^{-10}$$

$$\left(\frac{1}{9}\right)^{n+2} < 10^{-11}$$

$$(n+2) \log \frac{1}{9} < -11 \log 10$$

$$n > 9.5275\ldots$$

$$\therefore \quad n \geq 10$$

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Def (Taylor series)

Let $f(x)$ be infinitely differentiable (i.e. $f', f'', f''', \ldots$ all exist)

The Taylor series of $f(x)$ at $x=a$ is the infinite series

$$\boxed{\begin{aligned} T(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}}$$

Prop If the remainder term $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ in Taylor's thm $\to 0$ as $n \to \infty$ on an interval I, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ on I.

Example

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots$ for $|x| < 1$.
- $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1) x^k = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots$ for $|x| < 1$.
- $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ for $x \in \mathbb{R}$.
- $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for $x \in \mathbb{R}$.
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for $x \in \mathbb{R}$.
- $\ln x = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \cdots$ for $|x-1| < 1$.

Prop If $T(x)$ is the Taylor series of $f(x)$ centered at 0, then for any positive integer k, the Taylor series of $f(x^k)$ at 0 is $\boxed{T(x^k)}$.

Example

- $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^k x^{2k} + \cdots$ for $|x| < 1$.
- $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$ for $x \in \mathbb{R}$.

Prop (Addition / subtraction of Taylor series)

If the Taylor series of $f(x)$ and $g(x)$ about $x=c$ are

$$S(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

$$T(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots$$

then the Taylor series of $f \pm g$ about $x=c$ is

$$\boxed{(a_0 \pm b_0) + (a_1 \pm b_1)(x-c) + (a_2 \pm b_2)(x-c)^2 + \cdots}$$

Example For $x \in \mathbb{R}$,

$$\sin x + \cos x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$$

Prop (Multiplication of Taylor series)

If the Taylor series of $f(x)$ and $g(x)$ about $x=c$ are

$$S(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

$$T(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots$$

then the Taylor series of $f(x)g(x)$ about $x=c$ is

$$\boxed{\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(x-c)^n \\ &= a_0b_0 + (a_0b_1 + a_1b_0)(x-c) + (a_0b_2 + a_1b_1 + a_2b_0)(x-c)^2 \\ &\quad + \cdots\end{aligned}}$$

Proof $\dfrac{(fg)^{(n)}(c)}{n!} = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{f^{(k)}(c)\, g^{(n-k)}(c)}{n!}$ (Leibniz's rule)

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{f^{(k)}(c)\, g^{(n-k)}(c)}{n!}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} \cdot \frac{g^{(n-k)}(c)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k} \quad //$$

Example The Taylor series of $e^{4x} \ln(1+x)$ about 0 is

$$\left(1+4x+\frac{(4x)^2}{2!}+\frac{(4x)^3}{3!}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)$$

$$= x + \left(-\tfrac{1}{2}+4\right)x^2 + \left(\tfrac{1}{3}+4\cdot\left(-\tfrac{1}{2}\right)+8\right)x^3 + \cdots$$

$$= x + \frac{7x^2}{2} + \frac{19x^3}{3} + \cdots$$

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Example (Division)

Let $\sec x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

We have

$$1 = (\cos x)(\sec x)$$

$$= \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots\right)$$

$$= a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right)x^2 + \left(a_3 - \frac{a_1}{2}\right)x^3 + \left(a_4 - \frac{a_2}{2} + \frac{a_0}{24}\right)x^4 + \cdots$$

Comparing coefficients,

$$\begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_2 - \frac{a_0}{2} = 0 \Rightarrow a_2 = \frac{1}{2} \\ a_3 - \frac{a_1}{2} = 0 \Rightarrow a_3 = 0 \\ a_4 - \frac{a_2}{2} + \frac{a_0}{24} = 0 \Rightarrow a_4 = \frac{5}{24} \end{cases}$$

$$\therefore \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$$

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Prop (Composition of Taylor series)

If the Taylor series of $f(x)$ and $g(x)$ are $a_0+a_1x+a_2x^2+\dots$, $b_0+b_1x+b_2x^2+\dots$,

then the Taylor series of $g\circ f$ $(=g(f(x)))$ is

$$\boxed{b_0+b_1(a_0+a_1x+a_2x^2+\dots)+b_2(a_0+a_1x+a_2x^2+\dots)^2+\dots}$$

Example The Taylor series of $e^{\sin x}$ is

$$1+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)+\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)^2}{2!}+\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)^3}{3!}+\dots$$

$$=1+x+\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^5}{15}+\dots$$

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