MATH 1010F Lecture 18

2/11/2023

Last time

- Cauchy's mean value theorem:

  If f, g are continuous on $[a,b]$
  differentiable on (a,b)

  and $g(a) \neq g(b)$ and $g'(x) \neq 0$ on (a,b)

  then there exists $c \in (a,b)$ s.t.

  $$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

- L'Hopital's rule:

  Let $a \in \mathbb{R}$ or $= \pm\infty$, f, g differentiable near a, and:

  ① Both $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$
  or both $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

  ② $g'(x) \neq 0$ near a

  ③ $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or $= \pm\infty$

  Then $$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

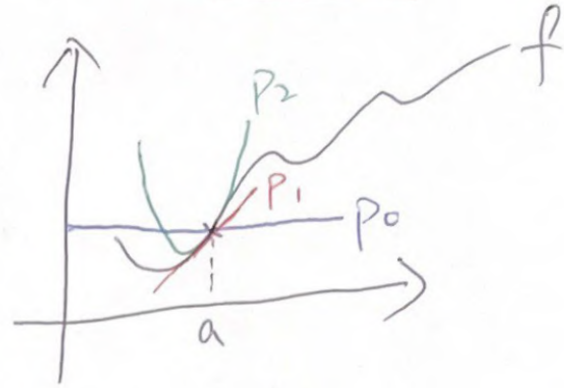
- Indeterminate forms:

  $$\underbrace{\frac{0}{0}, \frac{\pm\infty}{\pm\infty}}_{\text{directly use the rule}}, \underbrace{0\cdot(\pm\infty), \infty-\infty}_{\text{do some simplification, then use the rule}}, \underbrace{1^\infty, \infty^0, 0^0}_{\text{take log then use the rule}}$$

Taylor polynomials and Taylor series

Question: Given a function $f(x)$, can we approximate it near a point $x=a$ by a polynomial?

Consider a polynomial p_n of degree $\le n$:



$\boxed{n=0}$: $p_0(x)$ is a constant.

Take $p_0(x) = f(a)$.

Then we have $p_0(a) = f(a)$

(i.e. $p_0(x)$ and $f(x)$ have same value at a)

$\boxed{n=1}$: $p_1(x)$ is a linear polynomial.

Take $p_1(x) = f(a) + \underbrace{f'(a)}_{\text{slope of } f \text{ at } a}(x-a) \Rightarrow \begin{cases} p_1(a) = f(a) \\ p_1'(a) = f'(a) \end{cases}$

(i.e. $p_1(x)$ and $f(x)$ have same value and slope at a)

$\boxed{n=2}$: $p_2(x)$ is a quadratic polynomial.

Take $p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$$\Rightarrow \begin{cases} p_2(a) = f(a) + 0 + 0 = f(a) \\ p_2'(a) = f'(a) + f''(a)\cdot 0 = f'(a) \\ p_2''(a) = f''(a) \end{cases}$$

(i.e. $p_2(x)$ and $f(x)$ have same value, slope, concavity at a)

More generally, we have:

Def (Taylor polynomial)

Let f be n-times differentiable at a (i.e. $f(a), f'(a), f''(a), \ldots, f^{(n)}(a)$ exist).

We define the n^{th} order Taylor polynomial of $f(x)$ (also known as "Taylor polynomial of degree n") at $x=a$ to be

$$\boxed{\begin{aligned} p_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}}$$

Prop $p_n^{(k)}(a) = f^{(k)}(a)$ for all $0 \leq k \leq n$.

In other words, $p_n(x)$ is the "best" polynomial of degree $\leq n$ to approximate f near a.

Example Let $f(x) = \cos x$.

① Find the $(2n)^{th}$ order Taylor polynomial of f at 0.

② Approximate $f(0.1)$ using p_0, p_2, p_4.

Solution ① Note that
$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f''''(x) &= \cos x = f(x) \end{aligned}$$

$$\therefore f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k = 4m, \quad m \in \mathbb{Z} \\ -1 & \text{if } k = 4m+2, \quad m \in \mathbb{Z} \end{cases}$$

$$\therefore P_{2n}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + \frac{(-1)^n}{(2n)!}x^{2n}$$

② $P_0(x) = 1 \Rightarrow P_0(0.1) = 1$

$P_2(x) = 1 - \frac{1}{2!}x^2 \Rightarrow P_2(0.1) = 0.995$

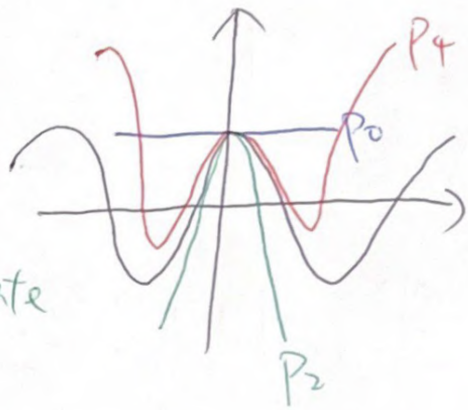
$P_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \Rightarrow P_4(0.1) = 0.995004166\ldots$

very close!

Actual value: $f(0.1) = \cos(0.1) = 0.99500416527\ldots$ //

Remarks

- Larger $n \Rightarrow P_n$ is a better approximation of f for x close enough to a



- The approximation may not be accurate at x far away from a!

Example Find the n^{th} order Taylor polynomial of $f(x) = \ln x$ at $a = 1$.

Solution

$f(x) = \ln x$ $\qquad f'''(x) = \frac{2}{x^3}$

$f'(x) = \frac{1}{x}$ $\qquad f''''(x) = -\frac{3 \cdot 2}{x^4}$

$f''(x) = -\frac{1}{x^2}$ $\qquad \vdots$

$$f^{(k)}(x) = (-1)^{k+1}\frac{(k-1)!}{x^k}$$

$$\therefore f^{(k)}(1) = \begin{cases} 0 & \text{if } k = 0 \\ (-1)^{k+1}(k-1)! & \text{if } k \geq 1 \end{cases}$$

$$\therefore\ p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$= 0 + \frac{(-1)^{1+1}(1-1)!}{1!}(x-1) + \frac{(-1)^{2+1}(2-1)!}{2!}(x-1)^2$$

$$+ \cdots + \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n \quad //$$

Question: How accurate is the approximation $f(x) \approx p_n(x)$?

Write $f(x) = \underbrace{p_n(x)}_{\text{Taylor polynomial}} + \underbrace{R_n(x)}_{\text{remainder / error term}}$

Thm (Taylor's theorem)

Let $x \neq a$ (i.e. $x > a$ or $x < a$).

Suppose $f^{(n)}$ exists and is continuous on $[a,x]$ (or $[x,a]$)

and $f^{(n+1)}$ exists on (a,x) (or (x,a)).

Then there exists $c \in (a,x)$ (or (x,a)) s.t.

$$f(x) = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{called remainder in Lagrange form}}$$