MATH 1010F Lecture 17 31/10/2023

Last time

- Lagrange's mean value theorem:

  If f is continuous on $[a,b]$
  and differentiable on (a,b),
  then there exists $c \in (a,b)$ s.t.
  $$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}.$$

- Increasing and decreasing functions:
  - f is increasing on (a,b) (i.e. $f(x) \le f(y)$ for all $x<y$)
    $\Leftrightarrow$ $f'(x) \ge 0$ on (a,b)
  - f is decreasing on (a,b) $\Leftrightarrow$ $f'(x) \le 0$ on (a,b)
  - f is strictly increasing ($f(x) < f(y)$ for all $x<y$)
    if $f'(x) > 0$ on (a,b)
  - f is strictly decreasing ($f(x) > f(y)$ for all $x<y$)
    if $f'(x) < 0$ on (a,b).

- Inequalities

Thm (Cauchy's mean value theorem)

If $f(x)$ and $g(x)$ are both continuous on $[a,b]$, differentiable on (a,b) and $\boxed{g(a) \neq g(b)}$ and $\boxed{g'(x) \neq 0}$ for any $x \in (a,b)$,
then there exists $c \in (a,b)$ s.t.
$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}.$$

Proof Let $h(x) = f(x) - \dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$.

Then we have:

$$h(a) = f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(a)-g(a)) = f(a) - 0 = f(a)$$

and

$$h(b) = f(b) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(b)-g(a)) = f(b) - (f(b)-f(a)) = f(a)$$

$\therefore h(a) = h(b)$.

Also, h is continuous on $[a,b]$ and differentiable on (a,b) as both f, g are cont. on $[a,b]$ and diff. on (a,b).

$\therefore$ By Rolle's theorem, there exists $c \in (a,b)$ s.t.

$$h'(c) = 0$$

note: $\left(h'(x) = \Big(f(x) - \underbrace{\frac{f(b)-f(a)}{g(b)-g(a)}}_{\text{constant}}(g(x) - \underbrace{g(a)}_{\text{const}})\Big)' \right)$

$= f'(x) - \dfrac{f(b)-f(a)}{g(b)-g(a)} g'(x)$

$$\therefore f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$$

$$\therefore \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \quad (\because g' \neq 0 \text{ on } (a,b))$$

//

Remark We cannot prove the Cauchy's MVT by applying the Lagrange's MVT to f and g separately!

We only have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}} = \frac{f'(c_1)}{g'(c_2)}$$

may be different!

Thm (L'Hopital's rule)

Let $a \in \mathbb{R}$ or $= \pm\infty$.

Suppose f, g are differentiable near a (i.e. on a small neighborhood of a), and all the following conditions are satisfied:

1. Both $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$

   or both $\lim\limits_{x \to a} f(x) = \pm\infty$ and $\lim\limits_{x \to a} g(x) = \pm\infty$

2. $g'(x) \neq 0$ near a

3. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists or $= \pm\infty$

Then $\boxed{\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}}$.

Remarks

- Similar results hold for one-sided limit

  $\left(\lim\limits_{x \to a^+} , \lim\limits_{x \to a^-} \right)$

- The L'Hopital's rule is very useful for computing limits in the form $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$ etc.

Proof (Mainly involves using Cauchy's MVT,

but there are many cases to consider

(e.g. $a \in \mathbb{R}$ vs $a = \pm\infty$, $\frac{0}{0}$ vs $\frac{\infty}{\infty}$ vs $\frac{\infty}{-\infty}$ ...)

We will only prove one of the cases.)

Consider the case $a \in \mathbb{R}$ and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

For any $x \neq a$, we apply Cauchy's MVT

to $f(x), g(x)$ on the interval between x and a (i.e. $[a,x]$ or $[x,a]$)

there exists c between x and a s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

Note: the theorem does not assume that $f(x), g(x)$ are defined at $x=a$

Here, we define $f(a) = g(a) = 0$ s.t. $f(x), g(x)$ are continuous at $x=a$

$$= \frac{f(x) - 0}{g(x) - 0}$$

$$\therefore \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c)}{g'(c)}$$

Now,

since c is between x and a,

as $x \to a$, c will also tend to a.

$$\therefore \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

Example

$$\lim_{x \to 1} \frac{x - e^{x-1}}{(x-1)^2} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{(x - e^{x-1})'}{((x-1)^2)'} \quad \text{(apply L'Hoptial's rule)}$$

$$= \lim_{x \to 1} \frac{1 - e^{x-1}}{2(x-1)} \quad \left(\tfrac{0}{0} \text{ form again}\right)$$

$$= \lim_{x\to 1} \frac{-e^{x-1}}{2}$$ (apply L'Hopital's rule once more)

$$= \frac{-e^{1-1}}{2} = -\frac{1}{2} \quad //$$

Example $\lim_{x\to\infty} \frac{e^{2x}}{x^2+4x+1}$ ($\frac{\infty}{\infty}$ form)

$$= \lim_{x\to\infty} \frac{(e^{2x})'}{(x^2+4x+1)'}$$ (apply L'Hopital's rule)

$$= \lim_{x\to\infty} \frac{2e^{2x}}{2x+4}$$ ($\frac{\infty}{\infty}$ form again)

$$= \lim_{x\to\infty} \frac{4e^{2x}}{2}$$ (apply L'Hopital's rule once more)

$$= +\infty \quad //$$

Example $\lim_{x\to\infty} \frac{\ln x}{x^2+3x+7}$ ($\frac{\infty}{\infty}$ form)

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{2x+3}$$

$$= \lim_{x\to\infty} \frac{1}{x(2x+3)}$$

$$= 0 \quad //$$

Example $\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3}$ ($\frac{0}{0}$ form)

$$= \lim_{x\to 0} \frac{\cos x - (\cos x - x\sin x)}{3x^2}$$

$$= \lim_{x\to 0} \frac{x\sin x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{3x} = \frac{1}{3} \quad //$$

Examples for which we <u>cannot</u> apply L'Hopital's rule:

① $\lim_{x \to \infty} \frac{\sin x + x}{x}$ ($\frac{\infty}{\infty}$ form)

$\times$ $\lim_{x \to \infty} \frac{\cos x + 1}{1}$ (the limit DNE, violating the third condition in the rule)

<u>Correct approach:</u> use squeeze thm

$$\frac{-1+x}{x} \le \frac{\sin x + x}{x} \le \frac{1+x}{x}$$

$$1 - \frac{1}{x} \le \frac{\sin x + x}{x} \le 1 + \frac{1}{x}$$

$$\lim_{x \to \infty} 1 - \frac{1}{x} = \lim_{x \to \infty} 1 + \frac{1}{x} = 1 \quad \therefore \lim_{x \to \infty} \frac{\sin x + x}{x} = 1 \quad //$$

② $\lim_{x \to 0} \frac{\sec x - 1}{e^{2x} - 1}$ ($\frac{0}{0}$ form)

$= \lim_{x \to 0} \frac{\sec x \tan x}{2e^{2x}}$ (can apply L'Hopital's rule, note: $(\sec x)' = \sec x \tan x$)

$\times$ $\lim_{x \to 0} \frac{\sec^2 x \tan x + \sec^3 x}{4e^{2x}} = \frac{1}{4}$

(Cannot apply L'Hopital's rule once more, as $\lim_{x \to 0} 2e^{2x} \neq 0, \pm\infty$, violating the first condition in the rule.)

<u>Correct approach:</u>

$$\lim_{x \to 0} \frac{\sec x - 1}{e^{2x} - 1} = \lim_{x \to 0} \frac{\sec x \tan x}{2e^{2x}} = \frac{0}{2 \cdot 1} = 0 \quad //$$

Standard forms in L'Hopital's rule: $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$

Other indeterminate forms : $0\cdot(\pm\infty)$, $\infty-\infty$, 1^∞, ∞^0, 0^0

Strategy : Convert them to $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$!

Example $\lim_{x\to 0^+} x\ln x$ ($0\cdot(-\infty)$ form)

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ ($\frac{-\infty}{\infty}$ form)

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$ (L'Hopital's rule)

$= \lim_{x\to 0^+} (-x)$ (direct simplification)

$= 0$ //

Remark : If we write $x\ln x = \frac{x}{\frac{1}{\ln x}}$ instead, the calculation becomes more difficult! Need to try different ways in general.

Example $\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x)$ ($\infty-\infty$ form)

$= \lim_{x\to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$

$= \lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to \frac{\pi}{2}} \frac{-\cos x}{-\sin x}$ (L'Hopital's rule)

$= \frac{-0}{-1} = 0$ //

<u>Example</u> $\lim_{x\to 0} (\cos x)^{\frac{1}{\sin x}}$ ($1^{\pm\infty}$ form)

Let $y = (\cos x)^{\frac{1}{\sin x}}$

$\ln y = \frac{1}{\sin x} \ln(\cos x)$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(\cos x)}{\sin x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot(-\sin x)}{\cos x}$ (L'Hopital's rule)

$= \frac{\frac{1}{1}\cdot(-0)}{1} = 0$

$\therefore \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\lim_{x\to 0}\ln y} = e^0 = 1$ //

($\because$ exponential function is continuous)

<u>Example</u> $\lim_{x\to\infty} x^{\frac{1}{x}}$ (∞^0 form)

Let $y = x^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln x$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$ (L'Hopital's rule)

$= 0$

$\therefore \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^0 = 1$ //

Example $\lim_{x \to 0^+} (1-\cos x)^{\frac{1}{\ln x}}$ (0^0 form)

Let $y = (1-\cos x)^{\frac{1}{\ln x}}$

$$\ln y = \frac{1}{\ln x} \ln(1-\cos x)$$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1-\cos x)}{\ln x} \quad \left(\frac{-\infty}{-\infty} \text{ form}\right)$$

$$= \lim_{x \to 0^+} \frac{\frac{\sin x}{1-\cos x}}{\frac{1}{x}} \quad (\text{L'Hopital's rule})$$

$$= \lim_{x \to 0^+} \frac{x \sin x}{1-\cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0^+} \frac{\sin x + x\cos x}{\sin x} \quad (\text{L'Hopital's rule again})$$

$$= \lim_{x \to 0^+} \left(1 + \frac{x}{\sin x} \cdot \cos x\right)$$

$$= 1 + (1)\cos(0) \quad \left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right)$$

$$= 2$$

$$\therefore \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^{\lim_{x \to 0^+} \ln y} = e^2 \quad //$$