MATH 1010 F Lecture 16 26/10/2023

Last time

- Finding global max./min. of a continuous function on a bounded interval
  - Check critical points and endpoints

- Curve sketching
  - Domain
  - x, y-intercepts
  - Asymptotes (vertical, horizontal, oblique)
  - critical points
  - inflection points.

- Rolle's theorem:

  If f is continuous on $[a,b]$, differentiable on (a,b) and $\boxed{f(a)=f(b)}$, then there exists $c \in (a,b)$ s.t.

$$\boxed{f'(c)=0}$$

- Lagrange's mean value theorem:

  If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ s.t.

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

Example Prove that

$$\tan^{-1}2022 + \frac{1}{1+2023^2} < \tan^{-1}2023 < \tan^{-1}2022 + \frac{1}{1+2022^2}.$$

Solution Let $f(x) = \tan^{-1}x$

Note that f is continuous on $[2022, 2023]$

and differentiable on $(2022, 2023)$.

By Lagrange's MVT, there exists $c \in (2022, 2023)$

s.t.

$$\frac{f(2023) - f(2022)}{2023 - 2022} = f'(c)$$

$$\therefore \frac{\tan^{-1}2023 - \tan^{-1}2022}{1} = \frac{1}{1+c^2} \quad \left(\begin{array}{l}\text{recall:}\\ (\tan^{-1}x)' = \frac{1}{1+x^2}\end{array}\right)$$

Now, since $2022 < c < 2023$, we have

$$\frac{1}{1+2023^2} < \frac{1}{1+c^2} < \frac{1}{1+2022^2}$$

$$\therefore \frac{1}{1+2023^2} < \tan^{-1}2023 - \tan^{-1}2022 < \frac{1}{1+2022^2}$$

$$\therefore \tan^{-1}2022 + \frac{1}{1+2023^2} < \tan^{-1}2023 < \tan^{-1}2022 + \frac{1}{1+2022^2}$$

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Def (increasing and decreasing function)

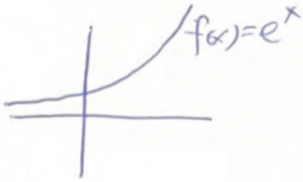
Let $f(x)$ be a function. We say that $f(x)$ is

- increasing (or monotonic increasing) if $f(x) \le f(y)$ for all $x<y$.
- decreasing (or monotonic decreasing) if $f(x) \ge f(y)$ for all $x<y$.

- **strictly increasing** if $f(x) < f(y)$ for all $x < y$.
- **strictly decreasing** if $f(x) > f(y)$ for all $x < y$.

Example

- $f(x) = e^x$ is strictly increasing on $\mathbb{R}$.



- $f(x) = \frac{1}{|x|}$ is $\begin{cases} \text{strictly decreasing on } (0,\infty) \\ \text{strictly increasing on } (-\infty,0) \end{cases}$

- $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$ is increasing on $\mathbb{R}$

Thm Let $f(x)$ be differentiable on (a,b). Then

① $f(x)$ is monotonic increasing on (a,b) if and only if
$f'(x) \geq 0$ for all $x \in (a,b)$.

② $f(x)$ is monotonic decreasing on (a,b) if and only if
$f'(x) \leq 0$ for all $x \in (a,b)$.

③ $f(x)$ is constant if and only if $f'(x) = 0$ for all $x \in (a,b)$.

④ $f(x)$ is strictly increasing if $f'(x) > 0$ for all $x \in (a,b)$.

⑤ $f(x)$ is strictly decreasing if $f'(x) < 0$ for all $x \in (a,b)$.

Note: They look trivial, but we have not proved them so far!

Proof ① $(\Rightarrow)$ If $f(x)$ is monotone increasing,
we have

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \geq 0 \quad (\because f(x+h)\geq f(x))$$

$\therefore$ $f'(x)\geq 0$ for all $x\in(a,b)$.

$(\Leftarrow)$ Suppose $f'(x)\geq 0$ for all $x\in(a,b)$.
For any $\alpha,\beta\in(a,b)$ with $\alpha<\beta$,
by Lagrange's MVT, there exists $c\in(\alpha,\beta)$
s.t.

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(c)$$

$$\therefore f(\beta)-f(\alpha) = f'(c)(\beta-\alpha)$$
$$\geq 0\cdot 0 = 0$$

$\therefore$ f is monotone increasing on (a,b)

② (Similar to ①)

③ by ① and ②

④ If $f'(x)>0$ for all $x\in(a,b)$,
then for any $\alpha,\beta\in(a,b)$ with $\alpha<\beta$,
by Lagrange's MVT there exists $c\in(\alpha,\beta)$ s.t.

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(c) > 0$$

$$\therefore f(\beta) > f(\alpha)$$

$\therefore$ f is strictly increasing on (a,b)

⑤ (Similar to ④)

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Remark • The converse of (4) is NOT true!

i.e. f is strictly increasing $\not\Rightarrow$ $f'(x) > 0$ for all $x \in (a,b)$.

e.g. $f(x) = x^3$ is strictly increasing

but $f'(0) = 0$ is not positive.

Example Prove that $1 - \frac{1}{x} \leq \ln x \leq x - 1$ for all $x \in (0, \infty)$.

Solution • To prove the first inequality $1 - \frac{1}{x} \leq \ln x$,

let $f(x) = \ln x - \left(1 - \frac{1}{x}\right)$

then $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$

Note that $f'(x) = 0 \Leftrightarrow x = 1$, and we have

	$0 < x < 1$	$x > 1$
$f'(x)$	$-$	$+$

$\therefore$ $f(x)$ attains its min. at $x = 1$

$\therefore$ $f(x) \geq f(1) = 0$ for all $x \in (0, \infty)$

$\Rightarrow$ $\ln x \geq 1 - \frac{1}{x}$ for all $x \in (0, \infty)$.

• To prove the second inequality $\ln x \leq x - 1$,

we let $g(x) = x - 1 - \ln x$

then $g'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$.

Note that $g'(x) = 0 \Leftrightarrow x = 1$, and

	$0<x<1$	$x>1$
$g'(x)$	$-$	$+$

$\therefore$ g attains its min. at $x=1$

$\therefore$ $g(x) \geq g(1)$ for all $x \in (0,\infty)$

$\therefore$ $x-1-\ln x \geq 0$

$x-1 \geq \ln x$ for all $x \in (0,\infty)$

$\therefore$ We conclude that $1-\frac{1}{x} \leq \ln x \leq x-1$ for all $x>0$ //