# MATH 1010 F Lecture 15 24/10/2023

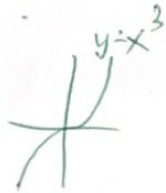
Last time :

- Application of differentiation : finding max / min
- Local maximum
  (exists $\delta > 0$ s.t. $f(x) \le f(a)$ for all $x \in (a-\delta, a+\delta)$)
- Local minimum
  (exists $\delta > 0$ s.t. $f(x) \ge f(a)$ for all $x \in (a-\delta, a+\delta)$)
- Critical points :
  $$f'(a) = 0 \text{ or DNE}$$
- Thm: If a continuous function $f$ has a local max / min at $x=a$, then $x=a$ must be a critical point.

  Remark: The converse is NOT true !

  $f(x) = x^3$, $f'(x) = 3x^2$

  $\therefore$ $x=0$ is a critical point but it is not a local max/min !




- First derivative test:
  Let $f$ be a continuous function and $x=a$ be a critical point.

  ① If there exists $\delta > 0$ s.t.
  $f'(x)$ changes sign from $-$ to $+$
  ($x \in (a-\delta, a)$, $x \in (a, a+\delta)$)
  then $x=a$ is a local min.

  ② If there exists $\delta > 0$ s.t.
  $f'(x)$ changes sign from $+$ to $-$
  ($x \in (a-\delta, a)$, $x \in (a, a+\delta)$)
  then $x=a$ is a local max.

Remark : • All such points at which $f'$ changes sign ($- \to +$ or $+ \to -$) are also known as turning points.

• $\{\text{turning points}\} \subseteq \{\text{critical points}\}$.

• e.g.  critical point but not turning point

• Second derivative test :

① If $f'(a)=0$ and $f''(a)<0$, then $f$ has a local max. at $x=a$

② If $f'(a)=0$ and $f''(a)>0$, then $f$ has a local min. at $x=a$.

Remark • can only be used if $f'(a)$ and $f''(a)$ exist!

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How can we find a global maximum / minimum in a bounded interval ?

Recall : (Extreme value theorem)

If $f$ is continuous on $[a,b]$, then $f$ has a global max/min on $[a,b]$.

Thm If a continuous function $f$ has a global maximum / minimum at $x=c \in [a,b]$, then either

① $x=c$ is a critical point (i.e. $f'(c)=0$ or DNE)

or

② $x=c$ is an endpoint (i.e. $c=a$ or $c=b$)

## Strategy for finding global max/min:

① Find all critical points and endpoints

② Compare the values of $f$ at those points to get the global max/min.

**Example** $f(x) = x^{\frac{5}{3}} + 2x^{\frac{2}{3}}$

Find the global max. and min. of $f$ on $[-1, 1]$.

**Sol** $f'(x) = \frac{5}{3}x^{\frac{2}{3}} + \frac{4}{3}x^{-\frac{1}{3}}$

- $f'(x)$ DNE $\Leftrightarrow$ $x = 0$
- $f'(x) = 0 \Leftrightarrow \frac{5}{3}x^{\frac{2}{3}} + \frac{4}{3}x^{-\frac{1}{3}} = 0$

$$\Leftrightarrow 5x + 4 = 0$$
$$\Leftrightarrow x = -\frac{4}{5}$$

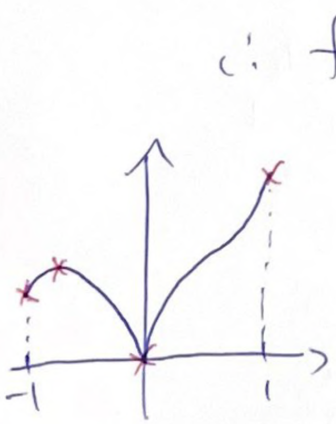
Check:

value of f at endpoints
$$\begin{cases} f(-1) = -1 + 2 = 1 \\ f(1) = 1 + 2 = 3 \end{cases}$$

value of f at critical points
$$\begin{cases} f(0) = 0 + 0 = 0 \\ f(-\frac{4}{5}) = (-\frac{4}{5})^{\frac{5}{3}} + 2(-\frac{4}{5})^{\frac{2}{3}} \approx 1.034 \end{cases}$$

$\therefore$ $f$ has global max at $x=1$ (with $f(1)=3$)

global min. at $x=0$ (with $f(0)=0$)



# Application of differentiation: Curve sketching

**Def** (concavity)

We say that $f(x)$ is

- Concave upward on $(a,b)$ if $f''(x) > 0$ on $(a,b)$.
- Concave downward on $(a,b)$ if $f''(x) < 0$ on $(a,b)$.

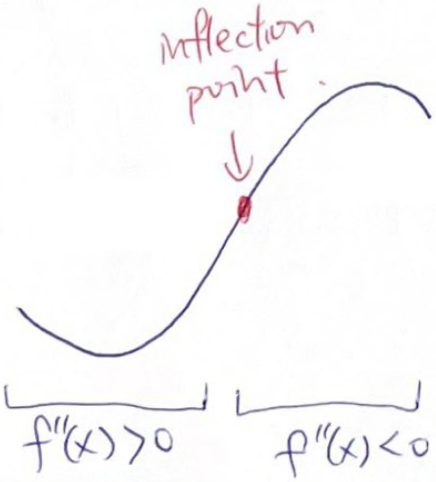
| | $f'(x) > 0$ | $f'(x) < 0$ |
|---|---|---|
| Concave upward ($f''(x) > 0$) |  |  |
| Concave downward ($f''(x) < 0$) |  |  |

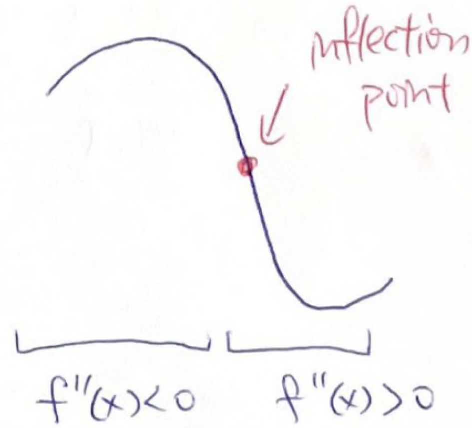
**Def** (inflection point)

We say that $x=a$ is an inflection point if $f''(x)$ changes sign at $x=a$.

(from $-$ to $+$ or from $+$ to $-$)

**Remark** If $f$ has an inflection point at $x=a$, then $f''(a) = 0$ or DNE.



## Def (Asymptotes)

- If $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

  then $y=b$ is a horizontal asymptote of $f(x)$.

- If $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

  then $x=a$ is a vertical asymptote of $f(x)$.

- If $\lim_{x\to\infty} \left(f(x) - (ax+b)\right) = 0$ or $\lim_{x\to-\infty} \left(f(x) - (ax+b)\right) = 0$,

  then $y = ax+b$ is an oblique asymptote of $f(x)$.

Example $f(x) = \dfrac{x^2-3x-4}{x-2}$

- vertical asymptote : $x=2$
- horizontal asymptote : $\lim_{x\to\pm\infty} f(x) = \pm\infty$

  $\therefore$ no horizontal asymptote.

- Oblique asymptote :

$$\frac{x^2-3x-4}{x-2} - (ax+b) = \frac{x^2-3x-4-(x-2)(ax+b)}{x-2}$$
$$= \frac{x^2-3x-4-(ax^2-2ax+bx-2b)}{x-2}$$
$$= \frac{(1-a)x^2+(2a-b-3)x+(2b-4)}{x-2}$$

for $\lim_{x\to\infty}\left(\frac{x^2-3x-4}{x-2} - (ax+b)\right) = 0$ or $\lim_{x\to-\infty}\left(\frac{x^2-3x-4}{x-2} - (ax+b)\right) = 0$,

$$\begin{cases} 1-a=0 \\ 2a-b-3=0 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=-1 \end{cases}$$

$\therefore$ $y = x-1$ is an oblique asymptote. //

Remark For $f(x) = \frac{\text{polynomial}}{\text{polynomial}}$, we can also use long division:

$$\begin{array}{r} x - 1 \\ x-2 \overline{) x^2 - 3x - 4} \\ \underline{x^2 - 2x} \\ -x - 4 \\ \underline{-x + 2} \\ -6 \end{array} \Rightarrow \frac{x^2-3x-4}{x-2} = \underbrace{x-1}_{\text{oblique asymptote}} + \frac{-6}{x-2}$$

$$\left( \because \frac{x^2-3x-4}{x-2} - (x-1) = \frac{-6}{x-2} \to 0 \right)$$

## Procedure for curve sketching

① Find:
- (natural) domain
- x-intercepts, y-intercepts
- asymptotes (vertical, horizontal, oblique)
- critical points
- inflection points

② sketch the curve based on the above information

Example $f(x) = x + \frac{1}{|x|}$

- Domain: $x \neq 0$
- x-intercept: $f(x) = 0 \Leftrightarrow x + \frac{1}{|x|} = 0 \Leftrightarrow x = -1$
- y-intercept: none $(\because f(0) \text{ DNE})$.
- vertical asymptote: $x = 0 \quad \left(\because \lim_{x\to 0^+} \left(x + \frac{1}{|x|}\right) = \infty\right)$
- horizontal asymptote: none $\left(\because \lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = -\infty\right)$
- Oblique asymptote: $y = x$

$$\left(\because f(x) - (ax+b) = (1-a)x + \frac{1}{|x|} - b \right.$$
$$\left. \lim_{x\to\pm\infty} (f(x) - (ax+b)) = 0 \Leftrightarrow \begin{cases} a = 1 \\ b = 0 \end{cases} \right)$$

· Critical points and inflection points :

If $x<0$, $f(x) = x - \frac{1}{x}$

$$f'(x) = 1 + \frac{1}{x^2} > 0$$

$\therefore$ no critical point.

$$f''(x) = -\frac{2}{x^3} > 0.$$

If $x>0$, $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$f'(x) = 0 \iff 1 - \frac{1}{x^2} = 0 \iff x = 1$$

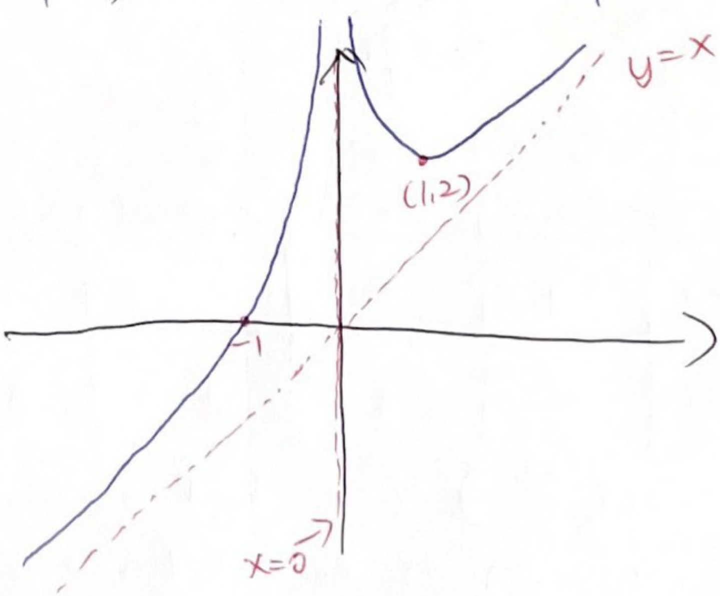
$\therefore$ $x=1$ is a critical point

$$f''(x) = \frac{2}{x^3} > 0$$

| | $x<0$ | $0<x<1$ | $x>1$ |
|---|---|---|---|
| $f'(x)$ | + | − | + |
| $f''(x)$ | + | + | + |

$\therefore$ $f(x)$ has a local minimum at $x=1$ (with $f(1) = 2$)

$f(x)$ has no inflection point.

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<u>Example</u> $f(x) = \dfrac{x^3}{(x-2)^2}$

- Domain : $x \neq 2$
- x-intercept : $f(x) = 0 \Leftrightarrow x = 0$
- y-intercept : $f(0) = 0$
- Vertical asymptote : $x = 2$
- Horizontal asymptote : none $\left(\because \lim_{x\to\pm\infty} f(x) = \pm\infty\right)$
- Oblique asymptote :

$$\frac{x^3}{(x-2)^2} - (ax+b) = \frac{x^3 - (ax+b)(x-2)^2}{(x-2)^2}$$

$$= \frac{(1-a)x^3 + (4a-b)x^2 + (-4a+4b)x - 4b}{(x-2)^2}$$

$$\therefore \lim_{x\to\pm\infty}\left(\frac{x^3}{(x-2)^2} - (ax+b)\right) = 0 \Leftrightarrow \begin{cases} 1-a=0 \\ 4a-b=0 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=4 \end{cases}$$

$\therefore$ $y = x+4$ is an oblique asymptote.

- Critical points and inflection points:

$$f'(x) = \frac{3x^2(x-2)^2 - x^3\cdot 2(x-2)}{(x-2)^4} = \frac{x^2(x-6)}{(x-2)^3}$$

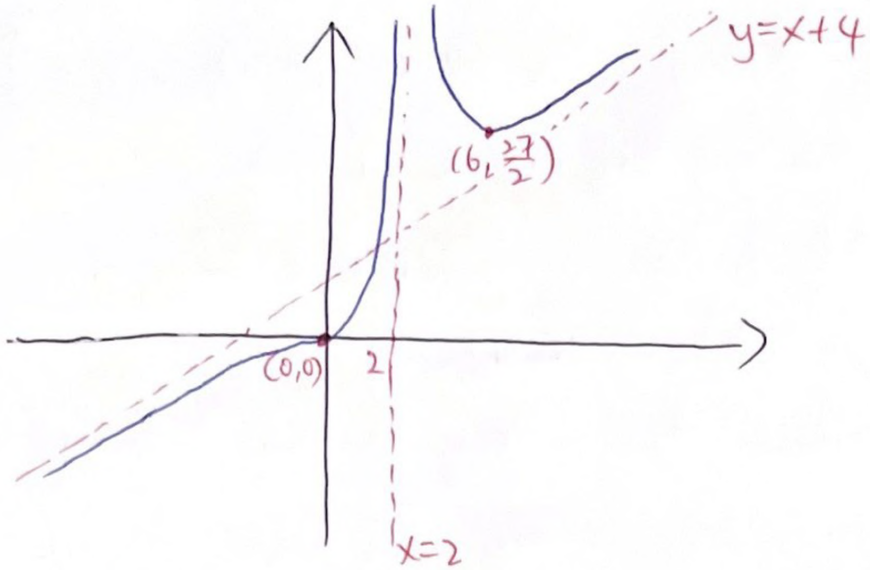
$\therefore$ $f'(x) = 0 \Leftrightarrow x = 0, 6$ are the critical points

$$f''(x) = \frac{24x}{(x-2)^4}$$

$f''(x) = 0 \Leftrightarrow x = 0$

| | $x<0$ | $0<x<2$ | $2<x<6$ | $x>6$ |
|---|---|---|---|---|
| $f'(x)$ | + | + | − | + |
| $f''(x)$ | − | + | + | + |

$\therefore$ $(6, \frac{27}{2})$ is a local minimum.
$(0, 0)$ is an inflection point.

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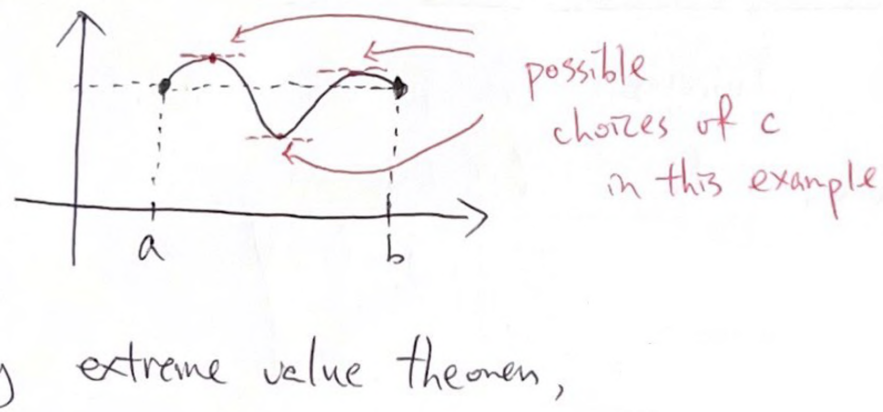
Mean value theorem (MVT)

Thm (Rolle's theorem)
Let $f$ be continuous on $[a,b]$ and differentiable on $(a,b)$.
If $\boxed{f(a)=f(b)}$, then there exists $c\in(a,b)$ s.t. $\boxed{f'(c)=0}$

values at the two endpoints




Proof By extreme value theorem,
$f$ has both global max. and min. on $[a,b]$

Case 1 If both max and min. occur at the endpoints $a,b$,
since $f(a)=f(b)$, we conclude that
$f$ is a constant function.

$\Rightarrow f'(x) = 0$ for all $x \in (a,b)$

$\therefore$ We can take any $c \in (a,b)$.

<u>Case 2</u> If $f$ has a global max / min at a point $c \in (a,b)$,

then $x=c$ is also a local max / min

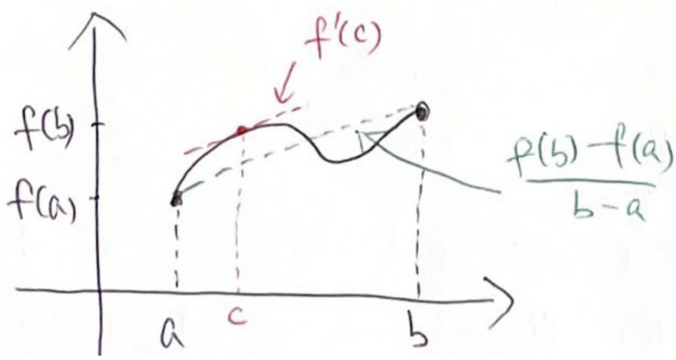
Thm last time $\Rightarrow f'(c) = 0$. //

<u>Thm</u> (<u>Lagrange's mean value theorem</u>)

Let $f$ be continuous on $[a,b]$ and differentiable on $(a,b)$.

Then there exists $c \in (a,b)$ s.t.

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$




<u>Proof</u> Let $g(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right)$

Then $g(a) = f(a) - (f(a) + 0) = 0$

$g(b) = f(b) - \left(f(a) + \frac{f(b)-f(a)}{b-a} \cdot (b-a)\right)$

$= f(b) - (f(a) + f(b) - f(a)) = 0$

$\because$ $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$,

$g$ is also cont. on $[a,b]$ and diff. on $(a,b)$.

$\therefore$ By Rolle's theorem,

there exists $c \in (a,b)$ s.t. $g'(c) = 0$

$$\therefore \quad f'(c) - \left(0 + \frac{f(b)-f(a)}{b-a}\right) = 0$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

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