MATH 1010F Lecture 14 19/10/2023

Last time
- Examples on n-times differentiability
- Review

Application of differentiation : Finding maximum / minimum

Recall :
- f has a local maximum (or "relative maximum") at $x=a$ if $f(x) \leq f(a)$ for all x near a
- f has a local minimum (or "relative minimum") at $x=a$ if $f(x) \geq f(a)$ for all x near a

More formally :

Def (Local maximum / minimum)

Let f be a continuous function.

We say $f(x)$ has a

① local maximum at $x=a$ if there exists $\delta > 0$ s.t. $f(x) \leq f(a)$ for all $x \in (a-\delta, a+\delta)$

(i.e. $a-\delta < x < a+\delta$)

② local minimum at $x=a$ if there exists $\delta > 0$ s.t. $f(x) \geq f(a)$ for all $x \in (a-\delta, a+\delta)$.

Thm Let $f(x)$ be a continuous function.

Suppose:

① $f(x)$ has a local maximum or local minimum at $x=a$

and

② f is differentiable at $x=a$

Then $f'(a)=0$.

Proof Suppose $f(x)$ has a local maximum at $x=a$.
(Proof for the other case is similar).

Then there exists $\delta>0$ s.t.

$$f(x) \le f(a) \quad \text{for all } x \in (a-\delta, a+\delta).$$

Now,

- For any $h<0$ with $a-\overset{\text{fixed}}{\delta} < a+h < a$,

  we have $f(a+h) \le f(a)$

  $\therefore f(a+h)-f(a) \le 0$

  $$\therefore \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \ge 0.$$

- For any $h>0$ with $a < a+h < a+\overset{\text{fixed}}{\delta}$,

  we have $f(a+h) \le f(a)$

  $$\therefore \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \le 0$$

Since f is differentiable at $x=a$,

$$\underbrace{\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}}_{\geq 0} = \underbrace{\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}}_{\leq 0} \quad (=f'(a))$$

$\therefore f'(a)=0$.

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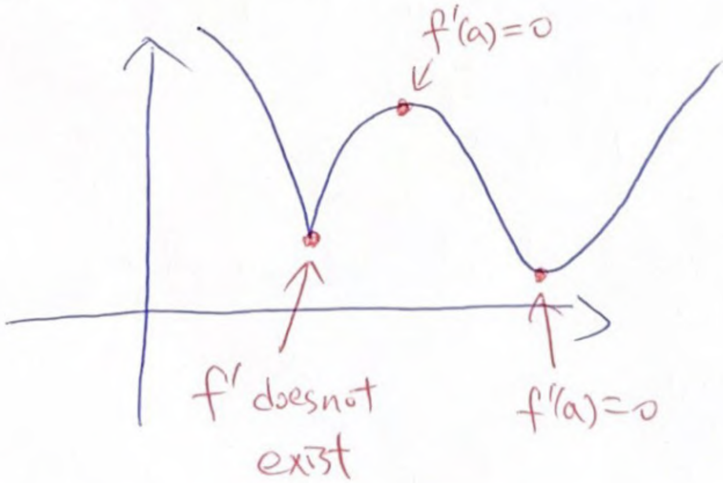
More generally, continuous functions may not be differentiable.
We have:

Thm Let $f(x)$ be a continuous function.
Suppose $f(x)$ has a local maximum or local minimum at $x=a$.
Then we have either:

① $f'(a)=0$ (i.e. $f'(a)$ exists and is equal to 0)

or

② $f'(x)$ does not exist at $x=a$.



Def (critical point)
$x=a$ is a critical point if $f'(a)=0$ or $f'(a)$ DNE

Thm (First derivative test)

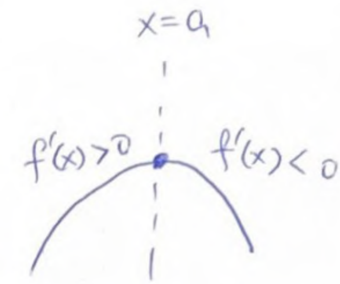
Let $f(x)$ be a continuous function and

$f'(a)=0$ or $f'(a)$ does not exist. (i.e. $x=a$ is a critical point)

Then:

① If there exists $\delta>0$ s.t.

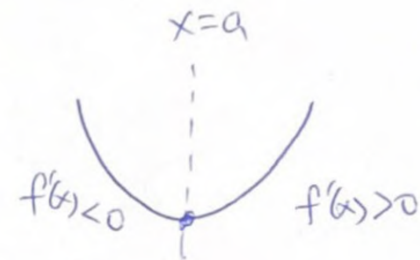
	$a-\delta<x<a$	$a<x<a+\delta$
$f'(x)$	$+$	$-$



then $f(x)$ has a local maximum at $x=a$.

② If there exists $\delta>0$ s.t.

	$a-\delta<x<a$	$a<x<a+\delta$
$f'(x)$	$-$	$+$



then $f(x)$ has a local minimum at $x=a$.

Thm (Second derivative test)

Let $f(x)$ be a differentiable function and $f'(a)=0$.

① If $f''(a)$ exists and <0,

then $f(x)$ has a local maximum at $x=a$.

② If $f''(a)$ exists and >0,

then $f(x)$ has a local minimum at $x=a$.

Remark : No conclusion if $f''(a) = 0$.

Example $f(x) = x^{\frac{5}{3}} + 2x^{\frac{2}{3}}$

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} + \frac{4}{3}x^{-\frac{1}{3}}$$

$\therefore$ We have

- $f'(x)$ does not exist if $x = 0$
- $f'(x) = 0 \iff \frac{5}{3}x^{\frac{2}{3}} + \frac{4}{3}x^{-\frac{1}{3}} = 0$

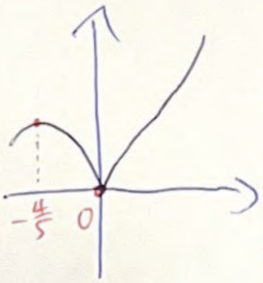
$$\iff 5x + 4 = 0 \iff x = -\frac{4}{5}$$

$\therefore$ The critical points are $x = 0, -\frac{4}{5}$.

Check :

	$x < -\frac{4}{5}$	$-\frac{4}{5} < x < 0$	$x > 0$
$f'(x)$	$+$	$-$	$+$

$\therefore$ By first derivative test,



$f(x)$ has a local maximum at $x = -\frac{4}{5}$

and a local minimum at $x = 0$.

For $x = -\frac{4}{5}$, we can also use the second derivative test:

$$f''\left(-\frac{4}{5}\right) = \frac{10}{9}\left(-\frac{4}{5}\right)^{-\frac{1}{3}} - \frac{4}{9}\left(-\frac{4}{5}\right)^{-\frac{4}{3}}$$

$$= -\frac{5\sqrt[3]{10}}{6} < 0$$

$\therefore$ $f(x)$ has a local max. at $x = -\frac{4}{5}$ //