MATH1010F University Mathematics

Review: Preliminary knowledge, limits of sequences, limits of functions, continuity, and differentiation

Gary Choi

October 17, 2023

https://www.math.cuhk.edu.hk/course/2324/math1010f

Quiz 1 reminder

- Date: October 19 (this Thursday)
- Time: 5:35PM - 6:20PM
- Venue for MATH1010F: TYW LT
  (T. Y. Wong Hall, 5/F, Ho Sin Hang Engineering Building)
- Closed book, closed notes
- List of approved calculators:
  `http://www.res.cuhk.edu.hk/images/content/examinations/use-of-calculators-during-course-examination/Use-of-Calculators-during-Course-Examinations.pdf`


Ho Sin Hang Engineering Building (SHB)
William M. W. Mong Engineering Building (ERB)
NOT Here
T. Y. Wong Hall (5/F)

Basic notations

Set: a collection of elements

- $\{a, b, c\}$ = a set containing three elements a, b, c
- $x \in A$ means "x is an element of the set A"
- $A \subset B$ (also written as $A \subseteq B$) means "A is a subset of B" (i.e. for any element $x \in A$, we have $x \in B$)
- $\{x : \cdots\} = \{x | \cdots\} = \{x \text{ such that } \cdots\}$
- $\mathbb{R}$ = the set of all real numbers
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ = the set of all integers
- $\mathbb{N} = \mathbb{Z}^+ = \{x \in \mathbb{Z} : x > 0\} = \{1, 2, 3, \ldots\}$ = the set of all positive integers
- $\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{p}{q} \text{ for some } p, q \in \mathbb{Z} \text{ with } q \neq 0\}$ = the set of all rational numbers
- $\emptyset = \{\ \}$ = empty set

Examples:

- $2 \in \mathbb{Z}$ (since 2 is an integer)
- $\pi \notin \mathbb{Q}$ (since π is an irrational number)
- $\{0, 2, 4, 6, \ldots\} \subset \mathbb{Z}$

Basic notations

- Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- Union of multiple sets $A_1, A_2, \ldots, A_n$:
$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$
- Intersection of multiple sets $A_1, A_2, \ldots, A_n$:
$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$
- Set difference: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

Examples:

- $\{1, 2, 3\} \cup \{1, 3, 4, 7\} = \{1, 2, 3, 4, 7\}$
- $\{1, 2, 3\} \cap \{1, 3, 4, 7\} = \{1, 3\}$
- $\{1, 2, 3\} \setminus \{1, 3, 4, 7\} = \{2\}$

Basic notations

Intervals:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ (open interval)
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval)
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, \infty) = \{x \in \mathbb{R} : x > a\}$
- $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$
- $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$
- $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

Examples:

- $(-1, 3) \cup (0, 4] = (-1, 4]$
- $[0, 5] \cap (1, \infty) = (1, 5]$
- $(0, 5) \setminus (1, 2) = (0, 1] \cup [2, 5)$
- $\bigcup_{n \in \mathbb{Z}} [2n\pi, (2n+1)\pi) = \cdots \cup [-2\pi, -\pi) \cup [0, \pi) \cup [2\pi, 3\pi) \cup \cdots$

(Lecture 1–2) Sequences

Examples:

- $a_n = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$
- $b_n = 2^{n-1} = 1, 2, 4, 8, \ldots$
- $c_n = (-1)^n = -1, 1, -1, 1, \ldots$
- Arithmetic sequences: $a_{n+1} - a_n = d$ for some constant d
- Geometric sequences: $a_{n+1} = ra_n$ for some constant r

Definitions:

- Monotonic increasing (or "increasing"): $a_n \leq a_{n+1}$ for all n
- Monotonic decreasing (or "decreasing"): $a_n \geq a_{n+1}$ for all n
- Monotonic: Either monotonic increasing or decreasing
- Strictly increasing: $a_n < a_{n+1}$ for all n
- Strictly decreasing: $a_n > a_{n+1}$ for all n
- Bounded below: there exists $M \in \mathbb{R}$ s.t. $a_n > M$ for all n
- Bounded above: there exists $M \in \mathbb{R}$ s.t. $a_n < M$ for all n
- Bounded: there exists $M \in \mathbb{R}$ s.t. $|a_n| < M$ for all n
  (i.e. both bounded below and bounded above)

(Lecture 1–2) Limits of sequences

Definitions:

- (Convergent sequence) If $\{a_n\}$ approaches a number L as n approaches infinity, we say $\lim_{n\to\infty} a_n = L$.
- (Divergent sequence) If no such L exists, we say that $\{a_n\}$ is divergent.
  Note: If $\lim_{n\to\infty} a_n = \infty$ or $-\infty$, it is also divergent.

Uniqueness of limit: If a_n is convergent, then the limit is unique.

Basic arithmetic rules: If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

- $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$
- $\lim_{n\to\infty} (ca_n) = ca$ (where c is a constant)
- $\lim_{n\to\infty} a_n b_n = ab$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ (if $b \neq 0$)

Example: $\lim_{n\to\infty} \left(\cos\frac{1}{n} - 2\left(\frac{3}{4}\right)^n + \frac{1}{n^2}\right) = 1 - 2\cdot 0 + 0 = 1$

(Lecture 1–2) Limits of sequences

Limits involving $\pm\infty$:

- $\infty \pm L = \infty$
- $-\infty \pm L = -\infty$
- $\infty + \infty = \infty$
- $-\infty - \infty = -\infty$
- $L \cdot \infty = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases}$
- $\frac{L}{\pm\infty} = 0$
- (Indeterminate forms) $\infty - \infty$, $\frac{\pm\infty}{\pm\infty}$, $\frac{0}{0}$, $0 \cdot \infty$: try further simplifying

Convergence $\Rightarrow$ **Boundedness**:

If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded.

Remark: The converse is **NOT** true, i.e. bounded $\not\Rightarrow$ convergent!
Example: $\{(-1)^n\} = -1, 1, -1, 1, \ldots$ is bounded but divergent.

(Lecture 2) Monotone convergence theorem

If $\{a_n\}$ is monotonic and bounded, then $\{a_n\}$ is convergent.

Other versions:

- If $\{a_n\}$ is monotonic increasing and bounded above, then $\{a_n\}$ is convergent.
- If $\{a_n\}$ is monotonic decreasing and bounded below, then $\{a_n\}$ is convergent.

Example: To prove that $\{a_n\}$ with $\begin{cases} a_{n+1} = \sqrt{a_n + 1} \\ a_1 = 1 \end{cases}$ is convergent, we prove that (i) $\{a_n\}$ is bounded by 2 (by MI) and (ii) $\{a_n\}$ is monotonic increasing.

Remark:
The converse is **NOT** true: convergent $\not\Rightarrow$ monotonic & bounded!
Example:
$\{\frac{(-1)^n}{n}\} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$ converges to 0, but the sequence is not monotonic.

(Lecture 3) Squeeze theorem (sandwich theorem)

> If $b_n \le a_n \le c_n$ for all n and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$,
> then $\lim_{n\to\infty} a_n = L$.

Example: $\lim_{n\to\infty} \frac{\sin(\cos n)}{n} = ?$

Solution: Since $-1 \le \sin(\cos n) \le 1$ for all n, we have

$$\frac{-1}{n} \le \frac{\sin(\cos n)}{n} \le \frac{1}{n}.$$

Now, since $\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$, by squeeze theorem, we have

$$\lim_{n\to\infty} \frac{\sin(\cos n)}{n} = 0.$$

Some possible ways to show that a sequence converges

(I) **Find the limit directly using some basic limit results**

- $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$, $\lim_{n\to\infty} \frac{1}{n} = 0$, ...

Example: $\lim_{n\to\infty}\left(\cos\frac{1}{n} + \left(\frac{3}{4}\right)^n + \frac{1}{n^2}\right) = 1 + 0 + 0 = 1$

(II) **Use the monotone convergence theorem**

- Show that the sequence is bounded and monotonic (may need to use mathematical induction)
- Conclude that the sequence converges (i.e. can write $\lim_{n\to\infty} a_n = L$, then solve some equations to find L if needed).

Example: Show that $\begin{cases} a_{n+1} = \sqrt{a_n + 1} \\ a_1 = 1 \end{cases}$ converges.

(III) **Use the squeeze theorem**

- Find b_n, c_n s.t. $b_n \le a_n \le c_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n \ (= L)$.
- Conclude that $\lim_{n\to\infty} a_n = L$.

Example: Show that $\{a_n\} = \left\{\frac{(-1)^n + \sin n}{n}\right\}$ converges.

If a way does not work, it **does NOT imply** that the sequence is divergent! Try another way.

Some possible ways to show that a sequence diverges

(I) **Show that $\{a_n\}$ is unbounded** (i.e. $\lim_{n\to\infty} |a_n| = \infty$)

- Reason: If a sequence converges, it must be bounded

Example $a_n = (-1)^n n^2$ diverges as $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n^2 = \infty$

(II) **Show that $\{a_n\}$ contains two subsequences which converge to two different values**

- Reason: If a sequence converges, then the limit must be unique

Example: $a_n = (-1)^n$ diverges since $\{a_1, a_3, a_5, \dots\}$ converges to -1 and $\{a_2, a_4, a_6, \dots\}$ converges to 1.

If a way does not work, it **does NOT imply** that the sequence is convergent! Try another way.

(Lecture 3) Infinite series

Series: $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$

Examples:

- $\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- (Arithmetic sum) $\sum_{k=1}^{n}(a + (k-1)d) = \frac{2a + (n-1)d}{2}$
- (Geometric sum) $\sum_{k=1}^{n} ar^{k-1} = \frac{a(r^n - 1)}{(r-1)}$ (if $r \neq 1$)

Convergence of infinite series: We say that an infinite series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ is convergent if the sequence of partial sums $\{s_n\}$ (where $s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$) converges.

Example: (Euler's number) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718$

(Lecture 3) Functions

Definitions:

- $f : A \to B$
  - A: Domain
  - B: Codomain
  - f: Some rule of assigning elements in A to elements in B
- Range of $f = \{f(x) : x \in A\}$ (also known as image of f)
- Natural domain = largest domain on which f can be defined

Examples:

- For $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, the range of f is $[0, \infty)$
- The natural domain of $f(x) = \frac{1}{\sqrt{x+1}}$ is $(-1, \infty)$
- The natural domain of $\tan(x)$ is

$$\mathbb{R} \setminus \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots\} = \bigcup_{n\in\mathbb{Z}} \left((n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi\right)$$

(Lecture 3–4) Injective, subjective, bijective functions, and inverse functions

- $f : A \to B$ is said to be injective (or "1-1", "one-to-one") if for any $x_1, x_2 \in A$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$ (Or equivalently, if $f(x_1) = f(x_2)$ then we have $x_1 = x_2$)
- $f : A \to B$ is said to be surjective (or "onto") if for any $y \in B$, there exists $x \in A$ such that $y = f(x)$
- f is bijective if it is both injective and surjective
- If $f : A \to B$ is a bijective function, the inverse function $f^{-1} : B \to A$ satisfies $f^{-1}(f(x)) = x$ for all $x \in A$ and $f(f^{-1}(y)) = y$ for all $y \in B$

Examples:

- $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$ is bijective
- $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is not injective as $f(-1) = f(1) = 1$
- $f : [0, \infty) \to \mathbb{R}$ with $f(x) = x^2$ is injective but not surjective
- $f : [0, \infty) \to [0, \infty)$ with $f(x) = x^2$ is bijective, and the inverse function is $f^{-1} : [0, \infty) \to [0, \infty)$ with $f^{-1}(y) = \sqrt{y}$

(Lecture 3–4) Even, odd, periodic functions

- f is an even function if $f(-x) = f(x)$ for all x
- f is an odd function if $f(-x) = -f(x)$ for all x
- f is a periodic function if there exists a constant k such that $f(x) = f(x+k)$ for all x

Examples:

- $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$ for all x
- $f(x) = x^3 + \sin x$ is odd because $f(-x) = (-x)^3 + \sin(-x) = -x^3 - \sin x = -(f(x))$ for all x
- $f(x) = x + 1$ is neither odd nor even because $f(-1) = 0 \neq \pm f(1)$
- $f(x) = 3\sin x + \cos\frac{x}{2}$ is periodic because $f(x+4\pi) = 3\sin(x+4\pi) + \cos\frac{x+4\pi}{2} = 3\sin(x+4\pi) + \cos\left(\frac{x}{2} + 2\pi\right) = 3\sin x + \cos\frac{x}{2} = f(x)$ for all x

(Lecture 4–5) Some common functions

Exponential function $e^x : \mathbb{R} \to \mathbb{R}^+$

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- bijective function

Logarithmic function $\ln : \mathbb{R}^+ \to \mathbb{R}$

- Inverse function of e^x ($y = e^x \Leftrightarrow x = \ln y$)
- bijective function

Sine function $\sin : \mathbb{R} \to [-1, 1]$

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- odd function (because $\sin(-x) = -\sin x$)
- periodic function (because $\sin(x + 2\pi) = \sin x$)

Cosine function $\cos : \mathbb{R} \to [-1, 1]$

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$
- even function (because $\cos(-x) = \cos x$)
- periodic function (because $\cos(x + 2\pi) = \cos x$)

(Lecture 4–5) Limit of functions

Definitions:

- Left-hand limit: We say that $\lim_{x\to a^-} f(x) = L$ if $f(x)$ is close enough to L whenever x is close enough to a and $x < a$.
- Right-hand limit: We say that $\lim_{x\to a^+} f(x) = L$ if $f(x)$ is close enough to L whenever x is close enough to a and $x > a$.
- Two-sided limit: We say that $\lim_{x\to a} f(x) = L$ if both the left-hand limit and the right-hand limit exist and are equal, i.e.

$$\boxed{\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L}$$

Remark: Whether f is defined at a or the value of f at a is **NOT** important for finding $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a} f(x)$

Example: If $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x^2 & \text{if } x > 0 \end{cases}$, we have

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$,
so the two-sided limit exists and we have $\lim_{x\to 0} f(x) = 0$ $(\neq 1)$

(Lecture 4–5) Properties of limits of functions

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

- $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$ (where c is a constant)
- $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (if $\lim_{x\to a} g(x) \neq 0$)

Examples:

- $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right) = \lim_{x\to 0} \frac{(x+1)-1}{x(x+1)} = \lim_{x\to 0} \frac{1}{x+1} = 1$
- $\lim_{x\to 2} \frac{2-x}{3-\sqrt{x^2+5}} = \lim_{x\to 2} \left(\frac{2-x}{3-\sqrt{x^2+5}} \cdot \frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}\right)$
  $= \lim_{x\to 2} \frac{(2-x)(3+\sqrt{x^2+5})}{4-x^2} = \lim_{x\to 2} \frac{3+\sqrt{x^2+5}}{2+x} = \frac{6}{4} = \frac{3}{2}$

(Lecture 4–5) Properties of limits of functions

Some other useful limit results:

- $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{x} = 1$
- $\displaystyle \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\displaystyle \lim_{x\to 0} \frac{\sin x}{x} = 1$

Examples:

- $\displaystyle \lim_{x\to 0} \frac{e^{3x} - 1}{x} = \lim_{x\to 0} \frac{e^{3x} - 1}{3x} \cdot 3 = 1 \cdot 3 = 3$
- $\displaystyle \lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \frac{\frac{\sin 2x}{2x}(2x)}{\frac{\sin 3x}{3x}(3x)} = \frac{\left(\lim_{x\to 0} \frac{\sin 2x}{2x}\right) \cdot 2}{\left(\lim_{x\to 0} \frac{\sin 3x}{3x}\right) \cdot 3} = \frac{1 \cdot 2}{1 \cdot 3} = \frac{2}{3}$

(Lecture 5) Sequential criterion

We have $\lim_{x\to a} f(x) = L$ (limit of function)

if and only if

For any sequence $\{x_n\}$ with $x_n \neq a$ for any n and $\lim_{n\to\infty} x_n = a$,

we have $\lim_{n\to\infty} f(x_n) = L$ (limit of sequence).

Consequence: If we can find two sequences $\{x_n\}, \{y_n\}$ such that:

- $x_n \neq a$, $y_n \neq a$ for all n and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$
- but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$,

then $\lim_{x\to a} f(x)$ does not exist.

Example: Prove that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Solution: Let $\{x_n\} = \left\{\frac{1}{n\pi}\right\} = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \cdots$ and $\{y_n\} = \left\{\frac{1}{2n\pi+\frac{\pi}{2}}\right\} = \frac{1}{2\pi+\frac{\pi}{2}}, \frac{1}{4\pi+\frac{\pi}{2}}, \frac{1}{6\pi+\frac{\pi}{2}}, \cdots$, then we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ but $\lim_{n\to\infty} f(x_n) = 0 \neq \lim_{n\to\infty} f(y_n) = 1$.

(Lecture 5) Squeeze theorem for functions

Let f, g, h be functions. If $f(x) \leq g(x) \leq h(x)$ for any $x \neq a$ on a neighborhood of a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then the limit of $g(x)$ at $x = a$ exists and we have $\lim_{x \to a} g(x) = L$.

Example: $\lim_{x \to 0} x \sin \dfrac{1}{e^{x^2} - 1} = ?$

Solution:

Since $-1 \leq \sin \frac{1}{e^{x^2}-1} \leq 1$ for all x, we have $-x \leq x \sin \dfrac{1}{e^{x^2} - 1} \leq x$.

As $\lim_{x \to 0}(-x) = 0 = \lim_{x \to 0} x$, by squeeze theorem, $\lim_{x \to 0} x \sin \dfrac{1}{e^{x^2} - 1} = 0$.

(Lecture 6–7) Limits at infinity

Definitions:

- We say that $\lim_{x\to\infty} f(x) = L$ if $f(x)$ is close enough to L whenever x is large enough.
- (Similar for $\lim_{x\to-\infty} f(x)$)

Examples:

- $\lim_{x\to\infty} \frac{1}{x-1} = 0$
- $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$
- $\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{3x} = \lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{3x\cdot\frac{2}{2}} =$

  $\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{2x\cdot\frac{3}{2}} = \left(\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{2x}\right)^{\frac{3}{2}} = e^{\frac{3}{2}}$
- $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$ and $\lim_{x\to\infty} \frac{(\ln x)^k}{x} = 0$ for any positive integer k

(Lecture 7) Asymptotes

- Horizontal asymptotes:
  If $\lim_{x\to\infty} f(x) = b$ **or** $\lim_{x\to-\infty} f(x) = b$,
  then $y = b$ is a horizontal asymptote of $f(x)$.

- Vertical asymptotes:
  If $\lim_{x\to a^-} f(x) = \pm\infty$ **or** $\lim_{x\to a^+} f(x) = \pm\infty$,
  then $x = a$ is a vertical asymptote of $f(x)$.

Examples:

- $y = 0$ is a horizontal asymptote of $f(x) = e^x$ since $\lim_{x\to-\infty} e^x = 0$
- $x = 2$ is a vertical asymptote of $f(x) = 1 + \dfrac{1}{x-2}$ since $\lim_{x\to 2^+} f(x) = \infty$

(Lecture 7) Continuity of functions

f is said to be continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

In other words, we have:

(i) The limit $\lim_{x \to a} f(x)$ exists (i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$), **and**

(ii) It is equal to the value of f at $x = a$.

f is said to be continuous on an interval (a, b) if f is continuous at every point on (a, b).

Examples:

- x^n, $\cos x$, $\sin x$, e^x are continuous on $\mathbb{R}$
- $\ln(x)$ is continuous on $\mathbb{R}^+$
- $f(x) = \begin{cases} -x + 1 & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$

(Lecture 7) Properties of continuous functions

Properties:

- If $f(x)$ and $g(x)$ are continuous at $x = a$, then the following functions are also continuous at $x = a$:
  - $f(x) \pm g(x)$
  - $cf(x)$ (where c is a constant)
  - $f(x)g(x)$
  - $\frac{f(x)}{g(x)}$ (if $g(a) \neq 0$)
- If $f(x)$ is continuous at $x = a$ and $g(u)$ is continuous at $u = f(a)$, then the composition $(g \circ f)(x)$ (i.e. $g(f(x))$) is also continuous at $x = a$.

Examples:

- $\cos(x) + 2x$ is continuous on $\mathbb{R}$ because both $\cos x$ and x are continuous on $\mathbb{R}$.
- $\sin(x^3 + 1)$ is continuous at $x = 0$ because $x^3 + 1$ is continuous at $x = 0$ and $\sin(u)$ is continuous at $u = 1$.

(Lecture 7) Intermediate value theorem and extreme value theorem

Intermediate value theorem (IVT):

> Let f be a continuous function on $[a, b]$.
> For any real number L between $f(a)$ and $f(b)$
> (i.e. $f(a) < L < f(b)$ or $f(b) < L < f(a)$),
> there exists $c \in (a, b)$ such that $f(c) = L$.

Example: Show that $f(x) = x^7 + x^3 + 1$ has a real root.
Solution: Note that $f(-1) = -1 < 0$ and $f(0) = 1 > 0$. As f is continuous, by IVT, there exists $c \in (-1, 0)$ s.t. $f(c) = 0$.

Extreme value theorem (EVT):

> Let f be a continuous function on $[a, b]$. Then there exists
> $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$
> (i.e. f has a global maximum and a global minimum in $[a, b]$).

(Lecture 8) Derivatives

f is said to be differentiable at $x = a$ if the following limit (called the derivative of f at $x = a$) exists:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Another form:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Remark: For piecewise functions, we need to check both

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \text{ and } \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

Example of finding derivative by definition (i.e. **first principle**):

- If $f(x) = x^2$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x.$$

(Lecture 8–9) Derivatives of common functions

- $(x^n)' = nx^{n-1}$
- $(e^x)' = e^x$
- $(\ln x)' = \frac{1}{x}$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x = \frac{1}{\cos^2 x}$
- $(c)' = 0$ (where c is a constant)
- $(\sinh x)' = \cosh x$ (where $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$)
- $(\cosh x)' = \sinh x$
- $(\tanh x)' = \text{sech}^2 x = \frac{1}{\cosh^2 x}$

(Lecture 8–9) Differentiation rules

If f and g are differentiable at a point, then the following functions are also differentiable at that point:

- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(cf(x))' = cf'(x)$ (where c is a constant)
- Product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

- Quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad (\text{if } g(x) \neq 0)$$

Examples:

- $(x^3 \sin x)' = (x^3)' \sin x + x^3(\sin x)' = 3x^2 \sin x + x^3 \cos x$
- $\left(\frac{\sin x}{x^2+1}\right)' = \frac{(\sin x)'(x^2+1)+(\sin x)(x^2+1)'}{(x^2+1)^2} = \frac{(x^2+1)\cos x+2x \sin x}{(x^2+1)^2}$

(Lecture 8–9) Differentiation rules

Chain rule:

If $f(x)$ is differentiable at $x = a$ and $g(u)$ is differentiable at $u = f(a)$, then $(g \circ f)$ is differentiable at $x = a$ and we have

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

In other words, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Examples:

- $(\sin x^2)' = \frac{d(\sin u)}{du}\frac{du}{dx}$ (let $u = x^2$) $= (\cos u)(2x) = 2x \cos x^2$
- $(e^{\sin x})' = e^{\sin x} \cos x$

A more complicated version: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

Example:

- $(\ln(\cos(x^3)))' = \dfrac{1}{\cos x^3} \cdot (-\sin(x^3)) \cdot (3x^2) = -3x^2 \tan x^3$

(Lecture 8–9) Continuity and differentiability

If f is differentiable at $x = a$, then f is continuous at $x = a$

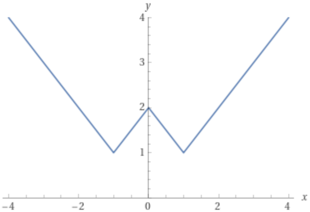
The converse is **NOT** true: if f is continuous at $x = a$, it may or may not be differentiable at $x = a$

Example: $f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

- $f(x)$ is continuous on $\mathbb{R}$ (i.e. at every point $x \in \mathbb{R}$):
  - For any $a < 0$, $\lim_{x \to a} f(x) = \lim_{x \to a}(-x) = -a = f(a)$
  - For any $a > 0$, $\lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a)$
  - For $a = 0$, we have $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(-x) = 0 = f(0)$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 = f(0)$, and hence $\lim_{x \to 0} f(x) = f(0)$
- $f(x)$ is not differentiable at $x = 0$:
  Note that $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$ but $\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$ and $\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{-h}{h} = -1$

Another example of continuous but not differentiable functions

$$f(x) = |x+1| - |x| + |x-1|$$
$$= \begin{cases} -(x+1) - (-x) - (x-1) & = -x & \text{if } x < -1 \\ (x+1) - (-x) - (x-1) & = x+2 & \text{if } -1 \leq x < 0 \\ (x+1) - (x) - (x-1) & = -x+2 & \text{if } 0 \leq x < 1 \\ (x+1) - (x) + (x-1) & = x & \text{if } x \geq 1 \end{cases}$$

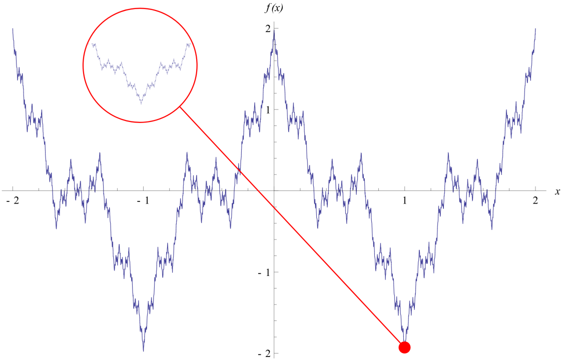


- $f(x)$ is continuous on $\mathbb{R}$
- $f(x)$ is not differentiable at $x = -1, 0, 1$

(Optional) Continuous but nowhere differentiable function

Weierstrass function (More details in MATH2050/2060)

`https://en.wikipedia.org/wiki/Weierstrass_function`



- $f(x)$ is continuous on $\mathbb{R}$
- $f(x)$ is not differentiable at any $x \in \mathbb{R}$

(Lecture 10–11) Implicit differentiation

Idea: Find y' without having to explicitly write $y = f(x)$.

Example: If $x \sin y + y^2 = x + 3y$, find the slope of tangent at $(0, 0)$.

$$
\begin{aligned}
(x \sin y + y^2)' &= (x + 3y)' \\
(\sin y + x(\cos y)y') + 2yy' &= 1 + 3y' \\
(x \cos y + 2y - 3)y' &= 1 - \sin y \\
y' &= \frac{1 - \sin y}{x \cos y + 2y - 3}
\end{aligned}
$$

The slope of tangent at $(0, 0)$ is $\dfrac{1 - \sin 0}{0 \cdot \cos 0 + 2 \cdot 0 - 3} = -\dfrac{1}{3}$

(Lecture 10–11) Logarithmic differentiation

Idea: Find the derivative of some complicated functions using logarithms.

Example: If $y = x^x$, find y'.

$$\begin{aligned} \ln y &= \ln(x^x) \\ (\ln y)' &= (x \ln x)' \\ \frac{1}{y} y' &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\ y' &= y(\ln x + 1) = x^x(\ln x + 1) \end{aligned}$$

(Lecture 10–11) Derivatives of some other special functions

- **More general exponential function**:
  Let $a > 0$ and define $a^x = e^{x \ln a}$. Then we have:
  - $a^{x+y} = a^x \cdot a^y$ for any $x, y \in \mathbb{R}$
  - $\lim_{x \to 0} \dfrac{a^x - 1}{x} = \ln a$
  - $(a^x)' = a^x \ln a$

  Example: $\left(2^{x^2+\cos x}\right)' = \left(2^{x^2+\cos x} \ln 2\right)(2x - \sin x)$
- **Inverse functions**:
  If $f(y)$ is a bijective and differentiable function with $f'(y) \neq 0$ for any y, then the inverse function $y = f^{-1}(x)$ is differentiable:

  $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

  Examples:

  $$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}, \ (\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}, \ (\tan^{-1} x)' = \frac{1}{1+x^2}$$

(Lecture 11–12) Higher order derivatives

- **Second derivative**:

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

- n-**th derivative**:

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\frac{dy}{dx}\right)\right)\right)$$

- 0-**th derivative**:

$$y^{(0)} = f^{(0)}(x) = f(x)$$

Examples:

- $(\sin x^2)'' = ((\sin x^2)')' = ((\cos x^2)(2x))'$
  $= (-\sin x^2)(2x)(2x) + 2\cos x^2 = -4x^2\sin x^2 + 2\cos x^2$
- Find y'' if $xy + \sin y = 1$:

$$(xy + \sin y)' = 1' \Rightarrow (y + xy' + y'\cos y) = 0 \Rightarrow y' = \frac{-y}{x + \cos y}$$

$$\Rightarrow y'' = -\frac{y'(x + \cos y) - y(1 - y'\sin y)}{(x + \cos y)^2} = \frac{2y(x + \cos y) + y^2\sin y}{(x + \cos y)^3}$$

(Lecture 11–12) Higher order differentiation rules

If f and g are n-times differentiable (i.e. $f^{(n)}$ and $g^{(n)}$ exist), then:

- $(f \pm g)^{(n)} = f^{(n)} \pm g^{(n)}$
- $(cf)^{(n)} = cf^{(n)}$ (where c is a constant)
- Leibniz's rule (product rule for higher order derivatives):

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

where $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$ is the binomial coefficient.

Example: $(x^3 \sin x)^{(4)}$
$= 1 \cdot (x^3)'''' \sin x + 4 \cdot (x^3)'''(\sin x)' + 6 \cdot (x^3)''(\sin x)'' + 4(x^3)'(\sin x)''' +$
$1 \cdot x^3 (\sin x)''''$
$= 0 + 24 \cos x - 36x \sin x - 12x^2 \cos x + x^3 \sin x$
$= (x^3 - 36x) \sin x + (24 - 12x^2) \cos x$

(Lecture 12–13) n-times differentiability and continuity

> If f is n-times differentiable at $x = a$
> ($f^{(n)}(a)$ exists, i.e. $f^{(n-1)}$ is differentiable at $x = a$),
> then $f^{(n-1)}$ is continuous at $x = a$.

f is n-times differentiable at $x = a$ (i.e. $f^{(n)}(a)$ exists)

$$\Downarrow$$

$f^{(n-1)}(a)$ exists and $f^{(n-1)}$ is continuous at $x = a$

$$\Downarrow$$

$$\vdots$$

$$\Downarrow$$

$f'(a)$ exists and f' is continuous at $x = a$

$$\Downarrow$$

f is continuous at $x = a$

However, the converse is **NOT** true!
Example: Let $f(x) = |x|x$, then:

- f is differentiable at $x = 0$
- f' is continuous at $x = 0$
- but f' is not differentiable at $x = 0$ (i.e. $f''(0)$ does not exist)