MATH1010F Lecture 13

17/10/2023

Last time

- Leibniz's rule:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

- f is n-times differentiable at $x=a$ (i.e. $f^{(n)}(a)$ exists)

$\Rightarrow f^{(n-1)}(x)$ exists and is continuous at $x=a$

$\not\Leftarrow$

Example Let $f(x) = |x|x$ for $x \in \mathbb{R}$. Show that:

① f is differentiable on $\mathbb{R}$.

② f' is continuous on $\mathbb{R}$.

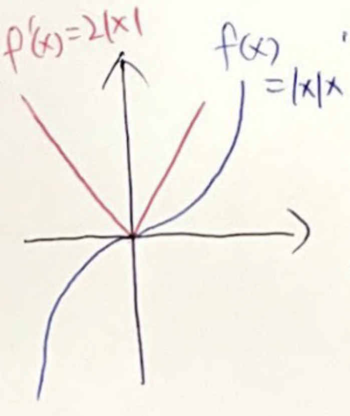
③ f' is not differentiable at $x=0$ (i.e. $f''(0)$ does not exist)

Solution ① Note that

$$f(x) = |x|x = \begin{cases} (-x)\cdot x = -x^2 & \text{if } x<0 \\ x \cdot x = x^2 & \text{if } x \geq 0 \end{cases}$$

- If $x<0$, $f'(x) = (-x^2)' = -2x$
- If $x>0$, $f'(x) = (x^2)' = 2x$
- If $x=0$,

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|h - 0}{h} = \lim_{h\to 0} |h| = 0$$



$\therefore$ f is differentiable on $\mathbb{R}$ and we have

$$f'(x) = \begin{cases} -2x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 2x & \text{if } x>0 \end{cases} = 2|x|.$$

② · For any $a<0$,

$$\lim_{x\to a} f'(x) = \lim_{x\to a} (-2x) = -2a = f'(a)$$ $\therefore f'$ is cont. at $x=a$

· For any $a>0$,

$$\lim_{x\to a} f'(x) = \lim_{x\to a} (2x) = 2a = f'(a)$$ $\therefore f'$ is cont. at $x=a$

· For $a=0$,

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (-2x) = -2\cdot 0 = 0 = f'(0)$$

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (2x) = 2\cdot 0 = 0 = f'(0)$$ $\therefore f'$ is cont. at $x=0$

$\therefore$ f' is continuous on $\mathbb{R}$.

③ $$f''(0) = \lim_{h\to 0} \frac{f'(0+h)-f'(0)}{h}$$ (by def.)

$$= \lim_{h\to 0} \frac{2|h|-0}{h}$$

$$= 2\lim_{h\to 0} \frac{|h|}{h}.$$

Note that

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

but $$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$\therefore$ $f''(0)$ does not exist.

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Example Let $f(x) = \begin{cases} x^4 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Show that

① f is twice differentiable on $\mathbb{R}$ (i.e. f'' exists on $\mathbb{R}$) but

② $f'''(0)$ does not exist.

Solution ①
- If $x \neq 0$, $f'(x) = 4x^3 \sin\frac{1}{x} + x^4\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$$= 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}$$

- If $x = 0$, $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$

$$= \lim_{h\to 0} \frac{h^4 \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} h^3 \sin\frac{1}{h}$$

$$= 0 \quad \left(\text{by squeeze thm, } \because \left|\sin\frac{1}{h}\right| \leq 1\right)$$

$$\therefore f'(x) = \begin{cases} 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

- If $x \neq 0$, $f''(x) = \left(12x^2 \sin\frac{1}{x} + 4x^3\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right) - \left(2x\cos\frac{1}{x} + x^2\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right)$

$$= (12x^2 - 1)\sin\frac{1}{x} - 6x\cos\frac{1}{x}$$

- If $x = 0$, $f''(0) = \lim_{h\to 0} \frac{f'(h) - f'(0)}{h}$

$$= \lim_{h\to 0} \frac{4h^3 \sin\frac{1}{h} - h^2 \cos\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} \left(4h^2 \sin\frac{1}{h} - h\cos\frac{1}{h}\right)$$

$= 0$ (by squeeze thm, $\because |\sin\frac{1}{h}| \le 1$ and $|\cos\frac{1}{h}| \le 1$)

$\therefore f$ is twice differentiable on $\mathbb{R}$ and

$$f''(x) = \begin{cases} (12x^2-1)\sin\frac{1}{x} - 6x\cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

② $f'''(0) = \lim_{h\to 0} \frac{f''(h) - f''(0)}{h}$ (by def.)

$$= \lim_{h\to 0} \frac{(12h^2-1)\sin\frac{1}{h} - 6h\cos\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} \left(12h\sin\frac{1}{h} - \frac{1}{h}\sin\frac{1}{h} - \cos\frac{1}{h}\right)$$

We can use the sequential criterion to show that the limit does not exist:

Let $a_n = \frac{1}{2n\pi}$
$b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, then $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$ but

$$\lim_{n\to\infty} \left(12a_n \sin\frac{1}{a_n} - \frac{1}{a_n}\sin\frac{1}{a_n} - \cos\frac{1}{a_n}\right)$$

$$= \lim_{n\to\infty} \left(12 \cdot \frac{1}{2n\pi} \cdot 0 - 2n\pi \cdot 0 - 1\right) = -1$$

while $\lim_{n\to\infty} \left(12b_n \sin\frac{1}{b_n} - \frac{1}{b_n}\sin\frac{1}{b_n} - \cos\frac{1}{b_n}\right)$

$$= \lim_{n\to\infty} \left(12 \cdot \frac{1}{2n\pi + \frac{\pi}{2}} \cdot 1 - \left(2n\pi + \frac{\pi}{2}\right) \cdot 1 - 0\right) = -\infty$$

$\therefore$ By sequential criterion,

$f'''(0)$ does not exist. //