MATH 1010 F Lecture 12 12/10/2023

Last time

- Implicit / logarithmic differentiation
- Derivatives of inverse functions
- Higher order derivatives

$$y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$y^{(n)} = \frac{d}{dx}\left(\frac{d}{dx}\left(\cdots \frac{dy}{dx}\right)\right)$$

Thm (Leibniz's rule) (Product rule for higher order derivatives)

If $f(x)$ and $g(x)$ are n-times differentiable functions (i.e. $f^{(n)}(x)$ and $g^{(n)}(x)$ exist), then

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Proof (By MI)

For $n=1$, $(fg)' = f'g + fg'$ (by product rule)

$\therefore$ True for $n=1$.

Assume it is true for n.

For $n+1$,

$$(fg)^{(n+1)} = \left(\left(fg\right)^{(n)}\right)' \quad \text{(by def)}$$

$$= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\right)' \quad \text{(by induction hypothesis)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(\left(f^{(n-k)}\right)' g^{(k)} + f^{(n-k)} \left(g^{(k)}\right)'\right) \quad \text{(by product rule)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(n-k+1)} g^{(k)} + f^{(n-k)} g^{(k+1)}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k+1)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n-(k-1))} g^{(k-1+1)}$$

(replacing k with k−1 so k starts with 1 now)

$$= \binom{n}{0} f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)}$$
$$+ \sum_{k=1}^{n} \binom{n}{k-1} f^{(n+1-k)} g^{(k)} + \binom{n}{n+1-1} f^{(0)} g^{(n+1)}$$

$$= f^{(n+1)} g^{(0)} + \left(\sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1}\right) f^{(n+1-k)} g^{(k)}\right) + f^{(0)} g^{(n+1)}$$

$$\left(\because \binom{n}{0} = 1 \text{ and } \binom{n}{n} = 1\right)$$

$$= f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(n+1-k)} g^{(k)} + f^{(0)} g^{(n+1)}$$

$$\left(\because \binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}\right.$$
$$\left.= \frac{n!(n-k+1+k)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}\right)$$

$$= \binom{n+1}{0} f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(n+1-k)} g^{(k)} + \binom{n+1}{n+1} f^{(0)} g^{(n+1)}$$

$$\left(\because \binom{n+1}{0} = \binom{n+1}{n+1} = 1\right)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(k)}$$

$\therefore$ By MI, the Leibniz's rule holds ! //

Example Let $y = x^2 e^{3x}$. Find $y^{(n)}$ for $n \geq 1$.

Solution Consider $f(x) = x^2$, $g(x) = e^{3x}$.

Note that :

- $f'(x) = 2x$, $f''(x) = 2$, $f''' = f'''' = \cdots = 0$
- $g'(x) = 3e^{3x}$, $g''(x) = 3^2 e^{3x}$, $\ldots$, $g^{(k)} = 3^k e^{3x}$.

$\therefore$ By Leibniz's rule,

- $n=1$: $y' = 2x\, e^{3x} + x^2 \cdot 3e^{3x} = (2x + 3x^2) e^{3x}$
- $n=2$: $y'' = 2e^{3x} + \binom{2}{1}(2x)(3e^{3x}) + x^2 \cdot 3^2 e^{3x}$

$$= (2 + 12x + 9x^2) e^{3x}$$

- $n \geq 3$: $y^{(n)} = \binom{n}{0} f^{(0)} g^{(n)} + \binom{n}{1} f^{(1)} g^{(n-1)} + \binom{n}{2} f^{(2)} g^{(n-2)}$

$$+ 0 + 0 + \cdots + 0 \quad (\because f^{(3)} = f^{(4)} = \cdots = 0)$$

$$= x^2 \cdot 3^n e^{3x} + n \cdot 2x \cdot 3^{n-1} e^{3x} + \frac{n(n-1)}{2} \cdot 2 \cdot 3^{n-2} \cdot e^{3x}$$

$$= (9x^2 + 6nx + n^2 - n) \cdot 3^{n-2} e^{3x}$$

//

Recall that we have:

$$f \text{ is differentiable at } x=a \Rightarrow f \text{ is continuous at } x=a$$

$$\not\Leftarrow \quad (\text{e.g. } f(x)=|x|)$$

Similarly, we have:

Prop If f is n-times differentiable at $x=a$
(i.e. $f^{(n)}(a)$ exists),
then $f^{(n-1)}(x)$ exists and is continuous at $x=a$.

$f^{(n)}$ exists at $x=a$ $\Rightarrow$ $f^{(n-1)}$ exists and is continuous at $x=a$ $\Rightarrow$ $f^{(n-2)}$ exists and is continuous at $x=a$ $\Rightarrow \cdots \Rightarrow$ f' exists and is continuous at $x=a$ $\Rightarrow$ f is continuous at $x=a$

$\not\Leftarrow \quad \not\Leftarrow \quad \not\Leftarrow \quad \not\Leftarrow \quad \not\Leftarrow$

__Example__
- x^m, e^x, $\cos x$, $\sin x$ are infinitely differentiable on $\mathbb{R}$
(i.e. $f^{(n)}$ exists for any n)
- $\ln(x)$ is infinitely differentiable on $(0, \infty)$

__Example__ Let $f(x) = |x|x$ for $x \in \mathbb{R}$. Show that:
1. f is differentiable on $\mathbb{R}$
2. f' is continuous on $\mathbb{R}$
3. f' is not differentiable at $x=0$ (i.e. $f''(0)$ does not exist).

__Next time__

Example Let $f(x) = \begin{cases} x^4 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Show that f is twice differentiable on $\mathbb{R}$ (i.e. f'' exists on $\mathbb{R}$) but $f'''(0)$ does not exist.

Next time