MATH 1010F Lecture 11

10/10/2023

Last time:
- Proof of chain rule
- $a^x = e^{x\ln a}$
- implicit differentiation

$F(x,y) = 0 \longrightarrow$ take derivative
$\longrightarrow$ rearrange terms
$\longrightarrow \frac{dy}{dx} = \dots$

Example Find $\frac{dy}{dx}$ if $y = x^x$, where $x > 0$.

Solution (Method 1)

$$\ln y = x \ln x$$
$$(\ln y)' = (x \ln x)'$$
$$\frac{1}{y} y' = x' \ln x + x (\ln x)'$$
$$\frac{1}{y} y' = \ln x + x \cdot \frac{1}{x}$$
$$y' = y(\ln x + 1)$$
$$= x^x(\ln x + 1)$$

(Method 2) $y = x^x = e^{x \ln x}$ $\quad (\because a^x = e^{x\ln a})$

$$\therefore \frac{dy}{dx} = (e^{x\ln x})'$$
$$= \frac{de^u}{du} \cdot \frac{du}{dx} \quad (\text{let } u = x\ln x)$$
$$= e^u \cdot (x' \ln x + x(\ln x)')$$

$$= e^u(\ln x + 1)$$

$$= e^{x\ln x}(\ln x+1)$$

$$= x^x(\ln x+1) \quad //$$

<u>Example</u> Let $u(x)$, $v(x)$ be differentiable functions of x, with $u(x) > 0$.

Find $\frac{d}{dx} u^v$.

<u>Solution</u>

$$\frac{d}{dx} u^v = \frac{d}{dx}\left(e^{v\ln u}\right) \qquad (\because a^x = e^{x\ln a})$$

$$= e^{v\ln u}\cdot(v\ln u)' \qquad (\text{chain rule})$$

$$= e^{v\ln u}\left(v'\ln u + v(\ln u)'\right) \qquad (\text{product rule})$$

$$= e^{v\ln u}\left(v'\ln u + v\cdot\frac{1}{u}\cdot u'\right)$$

$$= u^v v'\ln u + u^{v-1} v u' \quad //$$

Derivatives of inverse functions

Prop If $f(y)$ is a bijective and differentiable function with $f'(y) \neq 0$ for any y.

Then the inverse function $y = f^{-1}(x)$ is differentiable and

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}.$$

Proof By def of inverse function,

$$f(f^{-1}(x)) = x \quad \text{for any } x.$$

$$\therefore (f(f^{-1}(x))' = x'$$

(Chain rule) $\rightsquigarrow$ $$f'(f^{-1}(x)) \cdot ((f^{-1})'(x)) = 1$$

$$\therefore (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

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Prop (Derivatives of inverse of trigonometric functions)

① For $\sin^{-1} : (-1, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

(note: $\sin^{-1}$ is also known as arcsin $\neq \frac{1}{\sin x}$!)

② For $\cos^{-1} : (-1, 1) \to (0, \pi)$,

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

(note: $\cos^{-1} = \arccos$)

③ For $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

(note: $\tan^{-1} = \arctan$)

Proof ① $y = \sin^{-1} x$

$$\sin y = x$$

$$(\sin y)' = x'$$

$$(\cos y) \cdot y' = 1$$

$$\therefore y' = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

(note: $\cos y > 0$ for $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$)

$$= \frac{1}{\sqrt{1-x^2}}$$

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② $y = \cos^{-1} x$

$\cos y = x$

$(\cos y)' = x'$

$(-\sin y)\, y' = 1$

$\therefore\ y' = -\dfrac{1}{\sin y}$

$= -\dfrac{1}{\sqrt{1-\cos^2 y}}$

$= -\dfrac{1}{\sqrt{1-x^2}}$ //

③ $y = \tan^{-1} x$

$\tan y = x$

$(\tan y)' = x'$

$(\sec^2 y)\, y' = 1$

(Note: By quotient rule,

$(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)'$

$= \dfrac{(\sin x)'\cos x - (\sin x)(\cos x)'}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x} = \sec^2 x$)

$y' = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y}$ (by trigo. identity: $1+\tan^2 x = \sec^2 x$)

$= \dfrac{1}{1+x^2}$ //

Summary of common derivatives

- $(x^a)' = a x^{a-1}$
- $(e^x)' = e^x$
- $(\ln x)' = \dfrac{1}{x}$
- $(a^x)' = a^x \ln a$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x$
- $(c)' = 0$
- $(\sinh x)' = \cosh x$
- $(\cosh x)' = \sinh x$
- $(\tanh x)' = \text{sech}^2 x$
- $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}$
- $(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}}$
- $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$

Recall:

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\tanh x = \dfrac{\sinh x}{\cosh x}$

Summary of useful differentiation rules

- $(f \pm g)' = f' \pm g'$
- $(cf)' = cf'$
- $(fg)' = f'g + fg'$ (product rule)
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (if $g \neq 0$) (quotient rule)
- $(g \circ f)'(x) = g'(f(x))f'(x)$ (chain rule)
- implicit / logarithmic differentiation

Example $y = \sqrt{\frac{(x-1)^3(x-6)}{(x+7)^5}}$ with $x > 6$, find y'.

Solution:

$$\ln y = \frac{1}{2}\left(3\ln(x-1) + \ln(x-6) - 5\ln(x+7)\right)$$

$$\frac{1}{y}y' = \frac{1}{2}\left(\frac{3}{x-1} + \frac{1}{x-6} - \frac{5}{x+7}\right)$$

$$\therefore y' = \frac{y}{2}\left(\frac{3}{x-1} + \frac{1}{x-6} - \frac{5}{x+7}\right)$$

$$= \frac{1}{2}\sqrt{\frac{(x-1)^3(x-6)}{(x+7)^5}}\left(\frac{3}{x-1} + \frac{1}{x-6} - \frac{5}{x+7}\right) //$$

Example Find $\frac{dy}{dx}$ if $y = \tan^{-1}(\ln(\sin x + 2))$

Solution

$$\frac{dy}{dx} = \frac{d(\tan^{-1}u)}{du}\frac{du}{dx} \quad (\text{let } u = \ln(\sin x + 2))$$

$$= \frac{1}{1+u^2} \cdot \frac{d(\ln v)}{dv} \cdot \frac{dv}{dx} \quad (\text{let } v = \sin x + 2)$$

(i.e. we have $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$)

$$= \frac{1}{1+u^2} \cdot \frac{1}{v} \cdot (\cos x + 0)$$

$$= \frac{\cos x}{(1 + \ln^2(\sin x + 2))(\sin x + 2)} //$$

Example $\frac{d}{dx} \sin\sqrt{x^2+x+1} = ?$

By Chain rule,

$$\frac{d}{dx} \sin\sqrt{x^2+x+1} = \left(\cos\sqrt{x^2+x+1}\right) \cdot \frac{1}{2\sqrt{x^2+x+1}} \cdot (2x+1)$$

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$$\left(\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \right)$$

where $u = \sqrt{x^2+x+1}$

$v = x^2+x+1$

Higher order derivatives

Def

- Second derivative of $y = f(x)$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \left(\text{also denoted as } f''(x) \text{ or } y''\right)$$

- n^{th} derivative of $y = f(x)$ (where $n \geq 1$):

$$\begin{aligned} f^{(n)}(x) &= \frac{d^n y}{dx^n} \\ &= \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \\ &= \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots \frac{dy}{dx}\right)\right)\right) \end{aligned}$$

- 0^{th} derivative :

$$f^{(0)}(x) = f(x)$$

Example $y = x^2+3x+7$

$y' = 2x+3$

$y'' = 2$

$y^{(n)} = 0$ for $n \geq 3$.

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Example $f(x) = \sin x$

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f''''(x) = \sin x$
$= f(x)$

More generally,

$$f^{(n)}(x) = \begin{cases} \sin x & \text{if } n=4m \\ \cos x & \text{if } n=4m+1 \\ -\sin x & \text{if } n=4m+2 \\ -\cos x & \text{if } n=4m+3 \end{cases}$$, where m is a nonnegative integer

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Example Find y'' if $y = \ln(x^2+1)$

$y' = \frac{1}{x^2+1} \cdot 2x$ (chain rule)

$y'' = \frac{(2x)'(x^2+1) - (x^2+1)'(2x)}{(x^2+1)^2}$ (quotient rule)

$= \frac{2(x^2+1) - 4x^2}{(x^2+1)^2}$

$= \frac{2-2x^2}{(x^2+1)^2}$

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Example Find y'' if $x^2-y^2=1$, $y \neq 0$.

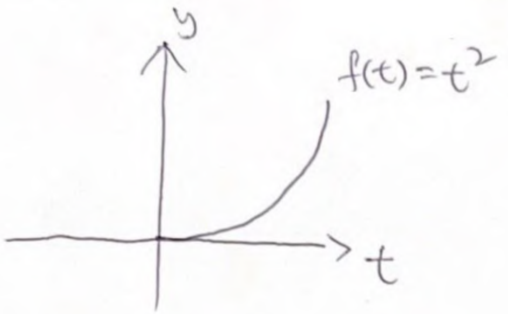
$(x^2-y^2)' = 1'$

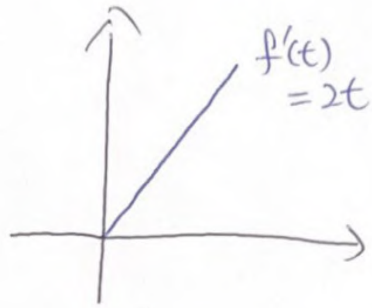
$2x - 2yy' = 0$

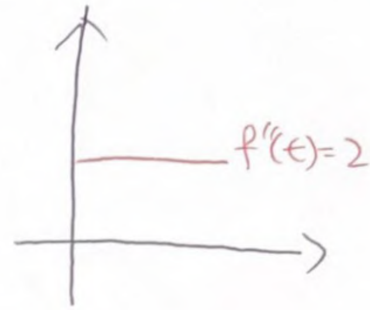
$\therefore y' = \frac{x}{y}$

$$\therefore \quad y'' = \left(\frac{x}{y}\right)'$$
$$= \frac{x'y - xy'}{y^2}$$
$$= \frac{y - x\left(\frac{x}{y}\right)}{y^2} \qquad \left(\because y' = \frac{x}{y}\right)$$
$$= \frac{y^2 - x^2}{y^3} = -\frac{1}{y^3} \quad //$$

Physical / geometric meaning of derivatives :







$f(t)$: a function representing the displacement (position) of a particle at time t

$f'(t)$: a function representing the velocity (rate of change of $f(t)$)

$f''(t)$: a function representing the acceleration (rate of change of $f'(t)$)

Product rule for higher order derivatives

$$(fg)' = f'g + fg'$$
$$(fg)'' = ((fg)')' = (f'g + fg')'$$
$$= (f'g)' + (fg')'$$
$$= (f''g + f'g') + (f'g' + fg'')$$
$$= f''g + 2f'g' + fg''$$

$$(fg)''' = (f''g + 2f'g' + fg'')'$$
$$= \cdots$$
$$= f'''g + 3f''g' + 3f'g'' + fg'''$$

More generally, we have:

Thm (Leibniz's rule)

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.