MATH 1010F Lecture 10

5/10/2023

Last time

- Derivatives of some common functions

  $(x^n)' = nx^{n-1}$, $(\ln x)' = \frac{1}{x}$, $(e^x)' = e^x$,

  $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(c)' = 0$

- Some useful rules:
  - $(f \pm g)' = f' \pm g'$
  - $(cf)' = cf'$
  - $(fg)' = f'g + fg'$ (Product rule)
  - $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ if $g(x) \neq 0$ (Quotient rule)
  - $(g \circ f)' = g'(f(x)) f'(x)$ (Chain rule)

    (another form: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$)

Proof of Chain rule: $(g \circ f)'(a) = g'(f(a)) f'(a)$

Idea: $$\frac{g(f(x)) - g(f(a))}{x-a} = \underbrace{\frac{g(f(x)) - g(f(a))}{f(x) - f(a)}}_{\substack{\text{should give us} \\ g'(f(a)) \\ \text{after taking } \lim_{x \to a}}} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\substack{\text{should give us} \\ f'(a) \\ \text{after taking } \lim_{x \to a}}}$$

Difficulty: If $f(x) - f(a) = 0$,

cannot directly divide by $f(x) - f(a)$!

Proof Define a function $F : \mathbb{R} \to \mathbb{R}$ as follows:

$$F(u) = \begin{cases} \dfrac{g(u) - g(f(a))}{u - f(a)} & \text{if } u \neq f(a) \\ g'(f(a)) & \text{if } u = f(a) \end{cases}$$

Note that:

- By construction, F is continuous at $u = f(a)$. ($\because$ g is differentiable at $f(a)$)
- f is continuous at $x = a$ ($\because$ f is differentiable at $x = a$)

$\therefore (F \circ f)$ is continuous at $x = a$
($\because$ Composition of continuous functions is continuous)

Now, for all $x \neq a$, we have

$$\frac{g(f(x)) - g(f(a))}{x - a} = F(f(x)) \cdot \frac{f(x) - f(a)}{x - a}$$

$\Bigg($ $\because$ if $f(x) \neq f(a)$, RHS $= \dfrac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \dfrac{f(x) - f(a)}{x - a} =$ LHS
if $f(x) = f(a)$, RHS $= g'(f(a)) \cdot 0 = 0 =$ LHS $\Bigg)$

$$\begin{aligned} \therefore (g \circ f)'(a) &= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} \\ &= \lim_{x \to a} \left(F(f(x)) \cdot \frac{f(x) - f(a)}{x - a} \right) \\ &= \lim_{x \to a} F(f(x)) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= F(f(a)) \cdot f'(a) = g'(f(a)) f'(a) \end{aligned}$$

$\because (F \circ f)$ is continuous

Def Let $a>0$. For $x \in \mathbb{R}$,

we define $a^x = e^{x \ln a}$

Prop Let $a>0$. We have:

① $a^{x+y} = a^x a^y$ for any $x, y \in \mathbb{R}$.

② $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$

③ $\frac{d}{dx} a^x = a^x \ln a$

Proof ① $a^{x+y} = e^{(x+y)\ln a}$ (by def)

$= e^{x\ln a + y \ln a}$

$= e^{x \ln a} \cdot e^{y \ln a}$ (by property of e^x)

$= a^x \cdot a^y$

//

② $\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{e^{x\ln a} - 1}{x}$

$= \lim_{x \to 0} \frac{e^{x \ln a} - 1}{x \ln a} \cdot \ln a$

$= (1) \cdot \ln a$ $\left(\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \right)$ proved previously

$= \ln a$

//

③ $\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$

$= \frac{de^u}{du} \cdot \frac{du}{dx}$ (where $u = x \ln a$) (chain rule)

$= e^u \cdot \ln a = e^{x \ln a} \ln a = a^x \ln a$ //

Def (implicit functions)

An implicit function is an equation of the form

$$F(x,y)=0.$$

Example

- $x^2+y^2-1=0$
- $x\sin y+y^2=0$

Implicit differentiation: Finding $\frac{dy}{dx}$ without explicitly finding $y(x)$.

Example Find $\frac{dy}{dx}$ if $x^2-xy-xy^2=0$.

Solution

$$x^2-xy-xy^2=0$$

$$(x^2-xy-xy^2)'=0$$

$$(x^2)'-(xy)'-(xy^2)'=0$$

(Product rule) $\quad 2x-(xy'+x'y)-(x'y^2+x(y^2)')=0$

(chain rule) $\quad 2x-(xy'+y)-(y^2+x\cdot 2y\cdot y')=0$

$$xy'+2xyy'=2x-y-y^2$$

$$y'=\frac{2x-y-y^2}{x+2xy} \quad //$$

In this case, okay to have y on RHS

Example Show that $\frac{d}{dx}(x^a) = a x^{a-1}$ for any $a \in \mathbb{R}$.
and $x > 0$

Solution Let $y = x^a$

Then $\ln y = a \ln x$

$$(\ln y)' = (a \ln x)'$$

$$\frac{1}{y} \cdot y' = a \cdot \frac{1}{x}$$

$$\therefore \quad y' = a \cdot \frac{y}{x} = a \cdot \frac{x^a}{x} = a x^{a-1} \quad //$$