MATH 1010F Lecture 9

3/10/2023

Last time

- Differentiable at $x=a$:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \left(= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \right)$$

exists.

(the derivative of f at $x=a$)

- Differentiable on (a,b): ← open interval (i.e. $\{x\in\mathbb{R} : a<x<b\}$)

f is differentiable at every point on (a,b)

- Differentiable at $x=a$ $\Rightarrow$ continuous at $x=a$

$$\not\Leftarrow$$

Derivatives of some common functions

Prop

① $\frac{d}{dx} x^n = n x^{n-1}$ (where $n\in\mathbb{Z}^+$) for all $x\in\mathbb{R}$

② $\frac{d}{dx} e^x = e^x$ for all $x\in\mathbb{R}$

③ $\frac{d}{dx} \ln x = \frac{1}{x}$ for all $x>0$.

④ $\frac{d}{dx} \cos x = -\sin x$ for all $x\in\mathbb{R}$

⑤ $\frac{d}{dx} \sin x = \cos x$ for all $x\in\mathbb{R}$

⑥ $\frac{d}{dx} c = 0$ (where c is a constant)

Remark ① is also true for general $a\in\mathbb{R}$:

$$\frac{d}{dx} x^a = a x^{a-1}$$

Proof ① $\frac{d}{dx} x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

$$= \lim_{h \to 0} \frac{(x+h-x)\left((x+h)^{n-1} + (x+h)^{n-2} \cdot x + \cdots + x^{n-1}\right)}{h}$$

$(\because a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}))$

$$= \lim_{h \to 0} \left((x+h)^{n-1} + (x+h)^{n-2} x + \cdots + x^{n-1}\right)$$

$$= (x+0)^{n-1} + (x+0)^{n-2} \cdot x + \cdots + x^{n-1}$$

$$= n x^{n-1}$$

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② $\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x$$

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$\left(\because \lim_{h \to 0} \frac{e^h - 1}{h} = 1, \text{ proved previously}\right)$

③ $\frac{d}{dx} \ln x = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$

$$= \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

$\left(\because \ln a - \ln b = \ln \frac{a}{b}\right)$

$$= \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x} \cdot x}$$

$$= \frac{1}{x} \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}}$$

$$= \frac{1}{x} \cdot 1$$

$\left(\because \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1, \text{ proved previously}\right)$

$$= \frac{1}{x}$$

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④ $\displaystyle \frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$

$\displaystyle = \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$ $\left(\because \cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}\right)$

$\displaystyle = \left(\lim_{h\to 0} -\sin\left(x+\frac{h}{2}\right)\right)\left(\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)$

$= (-\sin x)\cdot 1$ $\left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$

$= -\sin x$ //

⑤ $\displaystyle \frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$

$\displaystyle = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$ $\left(\because \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right)$

$\displaystyle = \left(\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\right)\left(\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)$

$= (\cos x)\cdot 1$ $\left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$

$= \cos x$ //

⑥ $\displaystyle \frac{d}{dx}c = \lim_{h\to 0}\frac{c-c}{h}$

$= 0$ //

Prop If $f(x)$ and $g(x)$ are differentiable at $x=a$, then $f\pm g$ and cf (where c is a constant) are also differentiable at $x=a$:

$(f\pm g)' = f' \pm g'$

$(cf)' = cf'$

Example $f(x) = 2e^x + 3\sin x - 4\cos x$

$$\begin{aligned} f'(x) &= (2e^x)' + (3\sin x)' - (4\cos x)' \\ &= 2(e^x)' + 3(\sin x)' - 4(\cos x)' \\ &= 2e^x + 3\cos x - 4(-\sin x) \\ &= 2e^x + 3\cos x + 4\sin x \end{aligned}$$

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Prop (Product rule)

If f and g are differentiable, then the product fg is also differentiable:

$$\boxed{(fg)' = f'g + fg'}$$

Proof

$$\begin{aligned} (fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right) \\ &= \lim_{h\to 0} \left(f(x+h) \cdot \underbrace{\frac{g(x+h) - g(x)}{h}} + g(x) \cdot \underbrace{\frac{f(x+h) - f(x)}{h}} \right) \\ &= f(x)\underbrace{g'(x)} + g(x)\underbrace{f'(x)} \end{aligned}$$

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Example $(x^2 \sin x)'$

$$\begin{aligned} &= x^2(\sin x)' + (x^2)'\sin x \\ &= x^2\cos x + 2x\sin x \end{aligned}$$

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Example Find $f'(x)$ if $f(x) = |x| \sin x$.

Solution $f(x) = |x| \sin x = \begin{cases} -x \sin x & \text{if } x < 0 \\ x \sin x & \text{if } x \geq 0. \end{cases}$

For $x < 0$, $f'(x) = (-x \sin x)'$

$$= -\left[(x)' \sin x + x (\sin x)'\right] \quad \text{(by product rule)}$$

$$= -(\sin x + x \cos x)$$

$$= -\sin x - x \cos x$$

For $x > 0$, $f'(x) = (x \sin x)'$

$$= \sin x + x \cos x$$

For $x = 0$, $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0}$

$$= \lim_{h \to 0} \frac{|h| \sin h - 0}{h}$$

$$= \lim_{h \to 0} |h| \cdot \frac{\sin h}{h}$$

$$= \left(\lim_{h \to 0} |h|\right) \left(\lim_{h \to 0} \frac{\sin h}{h}\right)$$

$$= 0 \cdot 1$$

$$= 0$$

$\therefore f'(x) = \begin{cases} -\sin x - x \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sin x + x \cos x & \text{if } x > 0 \end{cases}$ //

Prop (Quotient rule)

If f and g are differentiable and $g(x) \neq 0$,

then $\frac{f}{g}$ is also differentiable :

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

Proof

$$\left(\frac{f}{g}\right)' = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\, g(x)\, g(x+h)}$$

$$= \lim_{h \to 0} \left(\frac{f(x+h)g(x) - f(x)g(x)}{h\, g(x)\, g(x+h)} - \frac{f(x)g(x+h) - f(x)g(x)}{h\, g(x)\, g(x+h)} \right)$$

$$= \lim_{h \to 0} \left(\frac{g(x)}{g(x)g(x+h)} \cdot \frac{f(x+h) - f(x)}{h} - \frac{f(x)}{g(x)g(x+h)} \cdot \frac{g(x+h) - g(x)}{h} \right)$$

$$= \frac{gf'}{g^2} - \frac{fg'}{g^2}$$

$$= \frac{f'g - fg'}{g^2}$$

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Example $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$

$$= \frac{x'(x^2+1) - x(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(1)(x^2+1) - x(2x+0)}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2}$$

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Prop (Chain rule)

If f is differentiable at x

and g is differentiable at $f(x)$,

then $g \circ f$ is differentiable at x:

$$\boxed{(g \circ f)'(x) = g'(f(x))\, f'(x)}$$

Recall:

$(g \circ f)(x)$
$= g(f(x))$

Another form:

input x

intermediate variable $u = f(x)$

output $y = (g \circ f)(x) = g(f(x)) = g(u)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Example $\frac{d}{dx}(\sin x^2) = ?$

Solution Let $u = f(x) = x^2$, $g(u) = \sin u$

$$\frac{d}{dx}(\sin x^2) = \frac{d(\sin u)}{du} \cdot \frac{du}{dx}$$

$$= (\cos u) \cdot (2x)$$

$$= 2x \cos x^2$$

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Example $\frac{d}{dx}\left((1+x+x^2)^{10}\right)$

$$= \frac{d(u^{10})}{du} \cdot \frac{du}{dx} \qquad (\text{let } u = 1+x+x^2)$$

$$= 10u^9 \cdot (0+1+2x)$$

$$= 10(1+x+x^2)^9 \cdot (1+2x)$$

//

Example Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

① Find $f'(x)$.

② Is $f'(x)$ continuous at $x=0$?

Solution ① For $x \neq 0$,

$$f'(x) = \left(x^2 \sin\frac{1}{x}\right)'$$

$$= (x^2)' \sin\frac{1}{x} + x^2\left(\sin\frac{1}{x}\right)' \qquad (\text{product rule})$$

$$= 2x\sin\frac{1}{x} + x^2\left(\frac{d\sin u}{du} \cdot \frac{du}{dx}\right) \qquad (\text{let } u = \tfrac{1}{x} \text{ and use chain rule})$$

$$= 2x\sin\frac{1}{x} + x^2\left((\cos u) \cdot \left(\frac{-1}{x^2}\right)\right)$$

$$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

For $x=0$,

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}}{h} = \lim_{h\to 0} h\sin\frac{1}{h}$$

$\because \lim_{h\to 0} h = 0$ and $\left|\sin\frac{1}{h}\right| \leq 1$, squeeze thm $\Rightarrow f'(0) = 0$.

$$\therefore\ f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

② $$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$$

we can take two sequences $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$

s.t. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ but

$$\lim_{n\to\infty}\left(2x_n\sin\frac{1}{x_n} - \cos\frac{1}{x_n}\right) = \lim_{n\to\infty}\left(\frac{2}{2n\pi}\sin(2n\pi) - \cos(2n\pi)\right)$$
$$= 0 - 1 = -1$$

$$\lim_{n\to\infty}\left(2y_n\sin\frac{1}{y_n} - \cos\frac{1}{y_n}\right) = \lim_{n\to\infty}\left(\frac{2}{2n\pi + \frac{\pi}{2}}\sin\left(2n\pi + \frac{\pi}{2}\right) - \cos\left(2n\pi + \frac{\pi}{2}\right)\right)$$
$$= 0 - 0 = 0$$

$\therefore$ by sequential criterion,

$\lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$ does not exist

$\therefore$ $f'(x)$ is <u>not</u> continuous at $x=0$. //