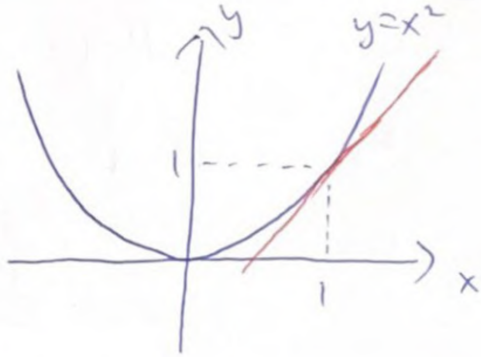
MATH 1010 F Lecture 8 28/9/2023

Last time

- Asymptotes
  - Horizontal $y=b$: $\lim_{x\to\infty} f(x)=b$ $\underline{\text{or}}$ $\lim_{x\to-\infty} f(x)=b$
  - Vertical $x=a$: $\lim_{x\to a^-} f(x)=\pm\infty$ $\underline{\text{or}}$ $\lim_{x\to a^+} f(x)=\pm\infty$
- Continuity
  - $\lim_{x\to a} f(x) = f(a)$
  - $+, -, \times, \div$, composition
- Intermediate value theorem
- Extreme value theorem

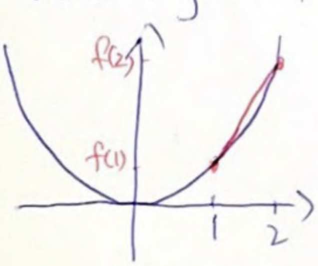
Differentiation



Example $f(x)=x^2$

Want to find the slope of tangent (i.e. rate of change) at $x=1$

We try different approximations:



$\text{slope} = \frac{f(2)-f(1)}{2-1} = 3$

$\text{slope} = \frac{f(1.1)-f(1)}{1.1-1} = 2.1$

$\to \dots \to$

better and better approximation

$$\text{slope} = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$

$$= 2$$

More generally,

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \text{instantaneous rate of change of } f \text{ at } x=a.$$

Remark: By a change of variable $x=a+h$,

we have $x\to a \iff h\to 0$

$$\therefore \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

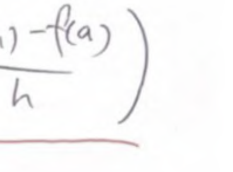
<u>Def</u> (Differentiable function)

- A function f is said to be <u>differentiable</u> at $x=a$ if the limit $\underline{f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \left(= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right)}$ exists. (sometimes also written as $\left.\frac{df}{dx}\right|_{x=a}$)

  $f'(a)$ is called the <u>derivative</u> of f at $x=a$.
- We say that $f(x)$ is differentiable in (a,b) if $f(x)$ is differentiable at every point in (a,b).

<u>Example</u> Let $f(x)=|x|$. Is f differentiable at $x=0$?

<u>Solution</u> Note that $f(x)=|x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$.



$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{-x-0}{x-0} = \lim_{x\to 0^-}(-1) = -1$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x-0}{x-0} = \lim_{x\to 0^+} 1 = 1 \neq -1$$

$\therefore \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist $\Rightarrow f$ is <u>not</u> differentiable at $x=0$. //

Example Let $f(x) = 2x^3 + 5$. Find $f'(1)$ from definition.

Sol

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{(2x^3 + 5) - 7}{x - 1}$$

$$= \lim_{x \to 1} \frac{2(x^3 - 1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{2(x-1)(x^2 + x + 1)}{x - 1}$$

recall: $\left(a^3 - b^3 = (a-b)(a^2 + ab + b^2)\right)$

$$= \lim_{x \to 1} 2(x^2 + x + 1)$$

$$= 2 \cdot (1 + 1 + 1) = 6$$

//

Example Let $f(x) = \frac{1}{\sqrt{x}}$ with $x > 0$. Find $f'(x)$.

Sol

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

$$= \frac{-1}{\sqrt{x} \cdot \sqrt{x} \cdot \left(\sqrt{x} + \sqrt{x}\right)} = -\frac{1}{2x^{\frac{3}{2}}}$$

//

Prop If f is differentiable at $x=a$, then f is continuous at $x=a$.

$$\boxed{\text{Differentiable at } x=a \Rightarrow \text{continuous at } x=a}$$

Proof If f is differentiable at $x=a$, then

$$\begin{aligned}\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right)\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right)\\ &= \underbrace{f'(a)}_{\text{exists}}\cdot 0 = 0\end{aligned}$$

$$\therefore \lim_{x\to a} f(x) = \lim_{x\to a} f(a) = f(a)$$

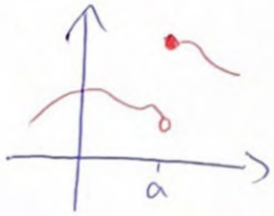
$\therefore$ f is continuous at $x=a$ //

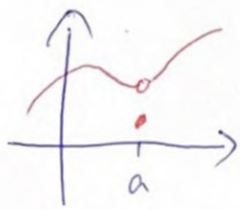
Remark The converse is not true:

$$\boxed{\text{Continuous at } x=a \not\Rightarrow \text{Differentiable at } x=a}$$

e.g. $f(x)=|x|$  is continuous at $x=0$ but not differentiable at $x=0$.

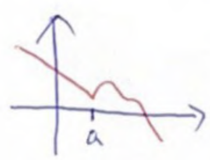
Example The following functions are not differentiable at $x=a$:

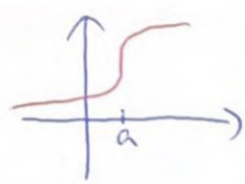




discontinuous at $x=a$

$\therefore$ not differentiable at $x=a$





$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ DNE

$\therefore$ not differentiable at $x=a$