MATH 1010F Lecture 7

26/9/2023

Last time
- Limits involving trigonometric functions
- Hyperbolic functions
- Limits at infinity $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$

Prop Let k be a positive integer.

① $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$

② $\lim_{x\to\infty} \frac{(\ln x)^k}{x} = 0$

Sol ① For any $x>0$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{k+1}}{(k+1)!} + \dots$$
$$> \frac{x^{k+1}}{(k+1)!}$$

$$\therefore \quad \frac{x^{k+1}}{e^x} < (k+1)!$$

$$0 < \frac{x^k}{e^x} < \frac{(k+1)!}{x}$$

just a constant, independent of x

Note that $\lim_{x\to\infty} \frac{(k+1)!}{x} = 0 = \lim_{x\to\infty} 0$

$\therefore$ By squeeze thm, $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$ //

② Let $x = e^y$, then we have $y = \ln x$.

Also, note that $x\to\infty$ as $y\to\infty$.

$\therefore \lim_{x\to\infty} \frac{(\ln x)^k}{x} = \lim_{y\to\infty} \frac{y^k}{e^y} = 0$ (by ①)

//

Def (Asymptotes)

- If $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to -\infty} f(x) = b$,
  then $y=b$ is a horizontal asymptote of $f(x)$.
- If $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$,
  then $x=a$ is a vertical asymptote of $f(x)$.

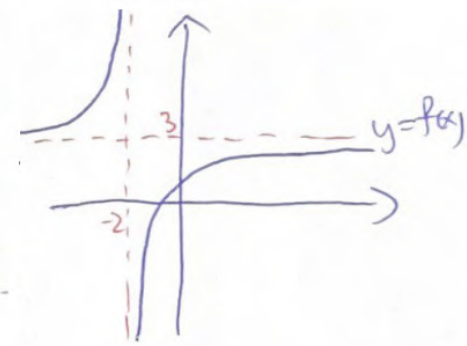
Example $f(x) = \dfrac{3x+5}{x+2} = \dfrac{3x+6-1}{x+2} = 3 - \dfrac{1}{x+2}$

Note that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(3 - \frac{1}{x+2}\right) = 3 - 0 = 3 = \lim_{x\to -\infty} f(x)$

$\therefore$ $y=3$ is a horizontal asymptote of $f(x)$.

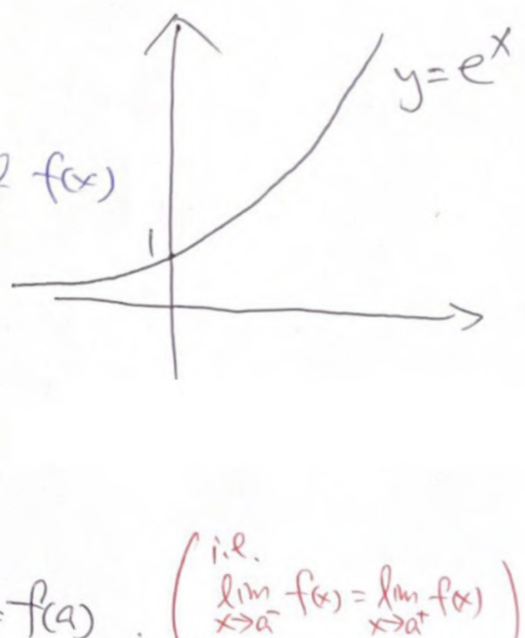
Also, $\lim_{x\to -2^-} f(x) = \infty$, $\lim_{x\to -2^+} f(x) = -\infty$

$\therefore$ $x=-2$ is a vertical asymptote of $f(x)$.



Example $f(x) = e^x$

$\lim_{x\to -\infty} f(x) = 0$ $\quad\therefore$ $y=0$ is a horizontal asymptote of $f(x)$

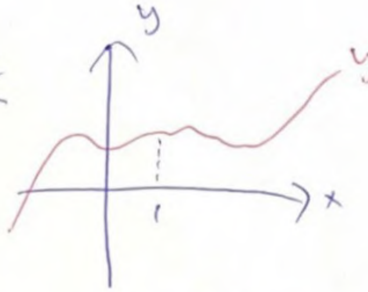


Continuity of functions

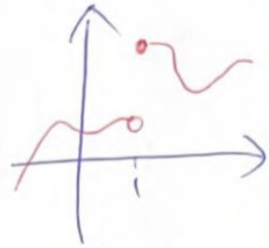
Def (Continuity)

- We say that a function $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$. $\left(\text{i.e. } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)\right)$
- We say that $f(x)$ is continuous on an interval in $\mathbb{R}$ if $f(x)$ is continuous at every point on it.

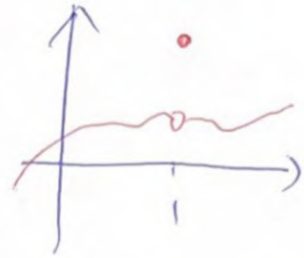
Example



$f(x)$ is continuous at $x=1$



$\lim_{x\to1} f(x)$ DNE (does not exist)



$\lim_{x\to1} f(x)$ exists but $\neq f(1)$

Discontinuous at $x=1$

Example $f:\mathbb{R}\to\mathbb{R}$ with

$$f(x)=\begin{cases} -x+1 & \text{if } x<0 \\ 1 & \text{if } x=0 \\ x^2+1 & \text{if } x>0 \end{cases}$$

Show that $f(x)$ is continuous at $x=0$.

Sol Note that $\lim_{x\to0^-} f(x) = \lim_{x\to0^-}(-x+1) = -0+1 = 1 = f(0)$

and $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}(x^2+1) = 0^2+1 = 1 = f(0)$

$\therefore \lim_{x\to0} f(x) = f(0)$

$\therefore f$ is continuous at $x=0$ //

Example Given that $f:\mathbb{R}\to\mathbb{R}$ with

$$f(x)=\begin{cases} 2x-1 & \text{if } x<2 \\ a & \text{if } x=2 \\ x^2+b & \text{if } x>2 \end{cases}$$

is continuous at $x=2$. Find the value of a and b.

Sol $\lim_{x\to2^-} f(x) = \lim_{x\to2^-}(2x-1) = 2\cdot2-1 = 3$

$\lim_{x\to2^+} f(x) = \lim_{x\to2^+}(x^2+b) = 2^2+b = 4+b$

$f(2) = a$

$\because f$ is continuous at $x=2$, we have $3=4+b=a \Rightarrow \begin{array}{l} a=3 \\ b=-1 \end{array}$ //

Prop

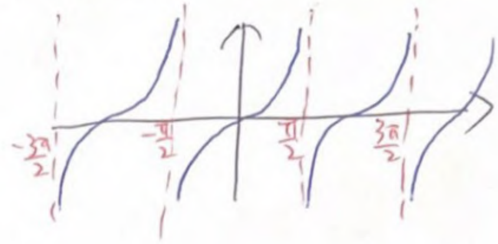
- x^n (for any positive integer n), $|x|$, e^x, $\cos x$, $\sin x$ are continuous on $\mathbb{R}$
- $\ln(x)$ is continuous on $\mathbb{R}^+$ (i.e. $\{x \in \mathbb{R} : x > 0\}$)
- $\tan(x)$ is continuous on its natural domain

(i.e. maximum domain on which $\tan(x)$ can be defined):

$$\mathbb{R} \setminus \{\pm\tfrac{\pi}{2}, \pm\tfrac{3\pi}{2}, \pm\tfrac{5\pi}{2}, \ldots\}$$

$\uparrow$

$\{x \in \mathbb{R}$ with $x \neq \pm\tfrac{\pi}{2}, \pm\tfrac{3\pi}{2}, \ldots\}$



Prop

If f, g are continuous at $x = a$, then the following functions are also continuous at $x = a$:

- $f(x) \pm g(x)$
- $c f(x)$ (where $c \in \mathbb{R}$)
- $f(x) g(x)$
- $\dfrac{f(x)}{g(x)}$ (if $g(a) \neq 0$)

Prop

If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$, then the composition

$$\underset{\displaystyle g(f(x))}{\underset{\shortparallel}{(g \circ f)(x)}}$$ is also continuous at $x = a$

Example $f(x) = e^x$ and $g(x) = \sin^2 x$ are continuous on $\mathbb{R}$

$\therefore$ $(f \circ g)(x) = e^{\sin^2 x}$ is also continuous on $\mathbb{R}$.

Example $\dfrac{\sin(\ln(\sqrt{x^2+1}))+x^2}{e^{\cos(x^2)-|x|}}$ is continuous on $\mathbb{R}$.

Prop If g is continuous and the limit of $f(x)$ at $x=a$ exists, then

f is $\underline{\underline{\text{not}}}$ required to be continuous

$$\lim_{x\to a}(g\circ f)(x)=\lim_{x\to a}g(f(x))=g\left(\lim_{x\to a}f(x)\right).$$

Example $\displaystyle\lim_{x\to\infty}\cos\left(\left(\frac{x+1}{x-1}\right)^x\right)=?$

Solution

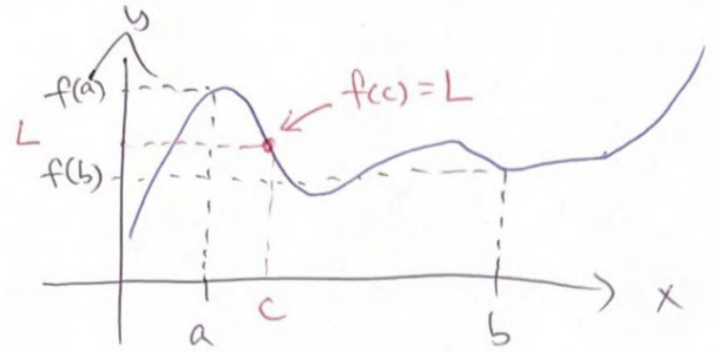
$$\begin{aligned}
&\lim_{x\to\infty}\cos\left(\left(\frac{x+1}{x-1}\right)^x\right)\\
&=\cos\left(\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x\right) \qquad \boxed{\because \cos \text{ is continuous}}\\
&=\cos\left(\lim_{x\to\infty}\left(\frac{x-1+2}{x-1}\right)^x\right)\\
&=\cos\left(\lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^x\right)\\
&=\cos\left(\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1+1}{2}\cdot 2}\right)\\
&=\cos\left(\lim_{x\to\infty}\left(\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}\cdot\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{1}{2}}\right)^2\right)\\
&=\cos\left(\left[\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}\cdot\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{1}{2}}\right]^2\right)\\
&=\cos\left(\left[e\cdot(1+0)^{\frac{1}{2}}\right]^2\right)\\
&=\cos e^2
\end{aligned}$$

$\boxed{\text{using } e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x}$

//

Thm (Intermediate value theorem)

Let f be a continuous function on $[a,b]$.
For any real number L between $f(a)$ and $f(b)$
(i.e. $f(a) < L < f(b)$ or $f(b) < L < f(a)$),
there exists $c \in (a,b)$ such that $f(c) = L$.



Example Show that $f(x) = x^7 + x^3 + 1$ has a real root
(i.e. $x^7 + x^3 + 1 = 0$ has a real solution).

Solution Note that
$$f(-1) = (-1)^7 + (-1)^3 + 1 = -1 - 1 + 1 = -1 < 0$$

$$f(1) = 1^7 + 1^3 + 1 = 1 + 1 + 1 = 3 > 0$$

$\therefore \; f(-1) < 0 < f(1)$

By IVT, there exists $c \in (-1, 1)$ such that
$f(c) = 0$

$\therefore \; f(x)$ has a real root

//

Example Show that $4^x - 3^x - 2^x = 1$ has a real solution.

Solution Let $f(x) = 4^x - 3^x - 2^x$

we have $f(1) = 4 - 3 - 2 = -1$

$f(2) = 4^2 - 3^2 - 2^2 = 3$

$\because f(1) = -1 < 1 < 3 = f(2)$

By IVT, there exists $c \in (1,2)$ s.t. $f(c) = 1$

i.e. $4^c - 3^c - 2^c = 1$

//

Def (Extremum)

We say that

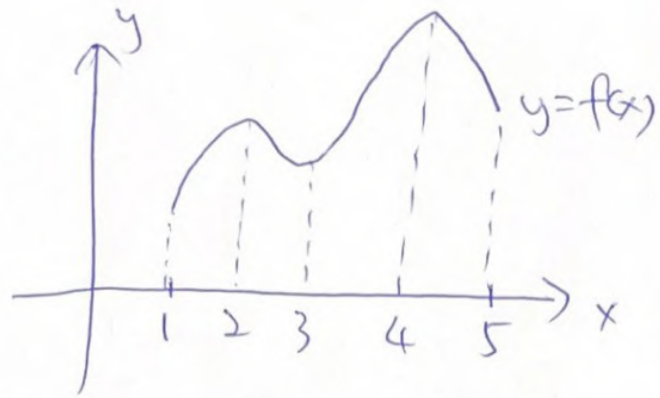
① f has a global maximum (or absolute maximum) at $x=a$ if $f(x) \le f(a)$ for all x in the domain of f.

② f has a global minimum (or absolute minimum) at $x=a$ if $f(x) \ge f(a)$ for all x in the domain of f

③ f has a local maximum (or relative maximum) at $x=a$ if $f(x) \le f(a)$ for all x near a.

④ f has a local minimum (or relative minimum) at $x=a$ if $f(x) \ge f(a)$ for all x near a.

Example



domain $= [1, 5]$

note: a global extremum is also a local extremum

f has a global maximum at $x = 4$

local maximum at $x = 2$ and $x = 4$

global minimum at $x = 1$

local minimum at $x = 1, 3, 5$.

Thm (Extreme value theorem)

Let f be a function which is continuous on a closed and bounded interval $[a, b]$.

Then there exists $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) \le f(x) \le f(\beta) \text{ for any } x \in [a, b].$$

In other words, f has a global maximum and a global minimum on $[a, b]$.