MATH 1010F Lecture 6 21/9/2023

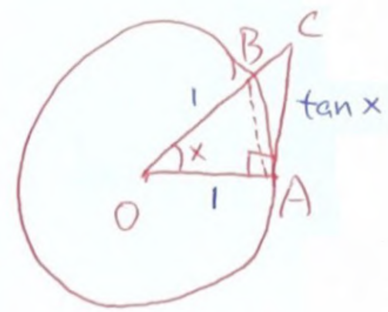
Last time

- Properties of limit of functions $(+,-,\times,\div)$
- Sequential criterion
- Squeeze thm for functions
- e^x, $\ln(x)$, $\cos x$, $\sin x$
- $\lim_{x\to 0}\frac{e^x-1}{x}=1$, $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$, $\lim_{x\to 0}\frac{\sin x}{x}=1$

Another proof for $\lim_{x\to 0}\frac{\sin x}{x}=1$ (Geometric)

① For $0<x<\frac{\pi}{2}$,

consider the unit circle



Note that

Area of sector OAB

$= \frac{x}{2\pi}\cdot$ (Area of the entire circle)

$= \frac{x}{2\pi}\cdot(1^2\pi) = \frac{x}{2}$

Also, note that

Area of $\triangle OAB \leq$ Area of sector OAB $\leq$ Area of $\triangle OAC$

$$\frac{1}{2}(1)(1)\sin x \leq \frac{x}{2} \leq \frac{1}{2}(\tan x)(1)$$

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2}$$

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

$\because \lim_{x\to 0^+} 1 = 1 = \lim_{x\to 0^+} \frac{1}{\cos x}$, by squeeze thm,

we have $\lim_{x\to 0^+} \frac{x}{\sin x} = 1$ $\therefore \lim_{x\to 0^+} \frac{\sin x}{x} = 1$.

② For $x<0$,

we let $y=-x$.

Then $x\to 0^- \Leftrightarrow y\to 0^+$.

$\because$ sin is an odd function

Now, $\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{y\to 0^+} \frac{\sin(-y)}{(-y)} = \lim_{y\to 0^+} \frac{-\sin y}{-y}$

$= \lim_{y\to 0^+} \frac{\sin y}{y} = 1$ (by ①)

$\therefore$ From ① and ②, we have $\lim_{x\to 0} \frac{\sin x}{x} = 1$ //

<u>Example</u> $\lim_{x\to 0} \frac{e^{5x}-e^{3x}}{\sin(4x)} = ?$

<u>Solution</u> $\lim_{x\to 0} \frac{e^{5x}-e^{3x}}{\sin(4x)} = \lim_{x\to 0} \frac{e^{3x}(e^{2x}-1)}{\sin(4x)}$

$$= \lim_{x\to 0} \frac{e^{3x}\cdot \frac{e^{2x}-1}{2x}\cdot 2\cancel{x}}{\frac{\sin(4x)}{4x}\cdot 4\cancel{x}}$$

$$= \frac{\left(\lim_{x\to 0} e^{3x}\right)\left(\lim_{x\to 0} \frac{e^{2x}-1}{2x}\right)\cdot 2}{\left(\lim_{x\to 0} \frac{\sin(4x)}{4x}\right)\cdot 4}$$

$$= \frac{1\cdot 1\cdot 2}{1\cdot 4} = \frac{2}{4} = \frac{1}{2}$$ //

Prop (Trigonometric identities)

1. $\cos^2 x + \sin^2 x = 1$
2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
3. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
4. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
5. $\sin(2x) = 2 \sin x \cos x$
6. $1 + \tan^2 x = \sec^2 x \; \left(= \frac{1}{\cos^2 x}\right)$

⋮

(see main lecture notes)

Example $\lim_{x \to 0} \frac{1 - \cos x}{x} = ?$

Solution

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

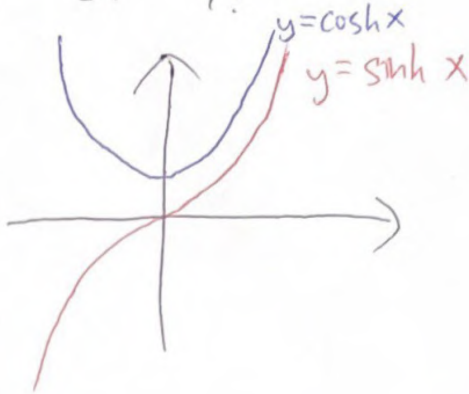
$$= \left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\lim_{x \to 0} \frac{\sin x}{1+\cos x}\right)$$

$$= 1 \cdot \frac{0}{2} = 0 \quad //$$

Def (Hyperbolic function)

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots,$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$



Note:
- $\cosh x$ is an even function
- $\sinh x$ is an odd function.

Prop (Hyperbolic identities)

① $\cosh^2 x - \sinh^2 x = 1$

②
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

③
- $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$
- $\sinh 2x = 2 \sinh x \cosh x$.

Example $\displaystyle \lim_{x \to 0} \frac{\cosh x - 1}{x} = ?$

Solution

$$\lim_{x \to 0} \frac{\cosh x - 1}{x} = \lim_{x \to 0} \frac{\frac{e^x + e^{-x}}{2} - 1}{x}$$

$$= \lim_{x \to 0} \frac{(e^x - 1) + (e^{-x} - 1)}{2x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{e^x - 1}{x} + \frac{1}{2} \lim_{x \to 0} \frac{e^{-x} - 1}{x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{e^x - 1}{x} - \frac{1}{2} \lim_{y \to 0} \frac{e^y - 1}{y} \quad (\text{let } y = -x)$$

$$= \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 1 = 0 \quad //$$

Def (Limits at infinity)

We say that $\lim_{x \to \infty} f(x) = L$ if

$f(x)$ is close enough to L whenever x is large enough.

(Similar for $\lim_{x \to -\infty} f(x)$)

Example $\lim_{x \to \infty} \frac{1}{x-1} = 0$, $\lim_{x \to -\infty} \frac{1}{x-1} = 0$

Prop $e = \lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$

limit of sequence

limit of function

Example $\lim_{x \to \infty} \left(1+\frac{2}{x}\right)^{3x}$

$$= \lim_{x \to \infty} \left(1+\frac{1}{\frac{x}{2}}\right)^{3x}$$

$$= \lim_{x \to \infty} \left(1+\frac{1}{\boxed{\frac{x}{2}}}\right)^{\boxed{\frac{x}{2}} \cdot 6}$$

same

$$= \left(\lim_{x \to \infty} \left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}}\right)^6$$

$$= e^6$$

//