MATH1010F Lecture 5 Supplementary Notes (Optional)

In Lecture 5, we talked about the proof of $\lim_{x\to 0} \frac{\sin x}{x} = 1$. In particular, using the infinite series form of $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \tag{1}$$

we have

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots. \tag{2}$$

We can then establish the following inequalities for $-1 < x < 1$ with $x \neq 0$ (see Lecture 5 notes for detail):

$$1 - \frac{x^2}{6} \leq \frac{\sin x}{x} \leq 1. \tag{3}$$

Since $\lim_{x\to 0}\left(1 - \frac{x^2}{6}\right) = 1 = \lim_{x\to 0} 1$, by the squeeze theorem, we can conclude that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Some students asked the following question: Can we directly take $\lim_{x\to 0}$ on both sides of Eq. (2) and conclude that

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} 1 - \lim_{x\to 0} \frac{x^2}{3!} + \lim_{x\to 0} \frac{x^4}{5!} - \lim_{x\to 0} \frac{x^6}{7!} + \cdots = 1 - 0 + 0 - 0 + \cdots = 1? \tag{4}$$

More generally, suppose $f(x) = \sum_{n=1}^{\infty} g_n(x)$, can we move a limit into the infinite summation (in other words, interchange the limit and the infinite summation):

$$\lim_{x\to a}\left(\sum_{n=1}^{\infty} g_n(x)\right) = \sum_{n=1}^{\infty}\left(\lim_{x\to a} g_n(x)\right)? \tag{5}$$

The answer is **NO** in general: While we can always interchange a limit and a *finite* summation (e.g. $\lim_{x\to a}(p(x) + q(x) + r(x)) = \lim_{x\to a} p(x) + \lim_{x\to a} q(x) + \lim_{x\to a} r(x))$ if the limits exist, interchanging a limit and an *infinite* summation is not always possible. The function $f(x) = \sum_{n=1}^{\infty} g_n(x)$ needs to satisfy certain additional properties to make Eq. (5) valid.

In fact, the study of the interchange of limits and infinite summations (or limits and integrals) is one of the major topics in mathematical analysis. Some relevant results include:

- If the partial sums $S_N = \sum_{n=1}^{N} g_n(x)$ are uniformly convergent, then Eq. (5) holds.
- The dominated convergence theorem.

These results are **out of the scope of MATH1010**, and so we did not use the argument in Eq. (4) in our proof of $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

A more detailed discussion on these theoretical aspects of limits can be found in MATH2050/2060 (Mathematical Analysis I/II).