MATH 1010F Lecture 5

19/9/2023

Last time

- Inverse function
- Even / odd functions
- Limit of functions
  - Left-hand limit $\lim_{x\to a^-} f(x)$
  - Right-hand limit $\lim_{x\to a^+} f(x)$
  - Two-sided limit $\lim_{x\to a} f(x)$

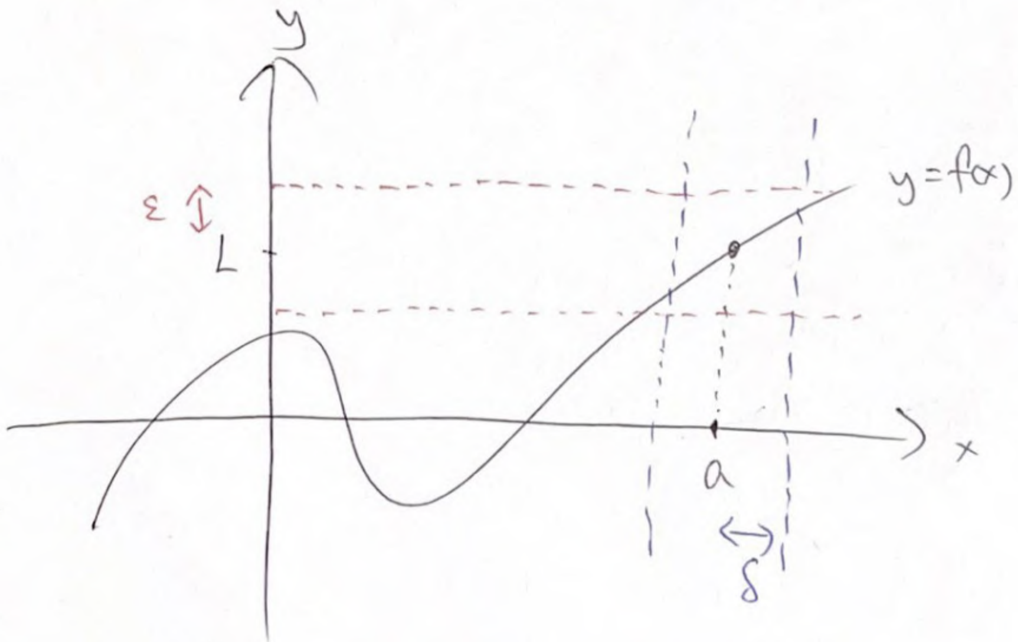
Def (Limit of functions) (formal, the ε-δ definition)

more in MATH 2050

We say that $\lim_{x\to a} f(x) = L$ if

for any $\varepsilon > 0$, there exists $\delta > 0$ s.t.

if $x \neq a$ and $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$.



Prop If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

① $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

② $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$ for any real number c

③ $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

④ $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Remark The properties also hold for $\lim_{x\to a^-}$ and $\lim_{x\to a^+}$

Example $\lim_{x\to 0} \left(\frac{1}{2x} - \frac{1}{x^2+2x} \right)$

$$= \lim_{x\to 0} \left(\frac{1}{2x} - \frac{1}{x(x+2)} \right)$$

$$= \lim_{x\to 0} \frac{x+2-2}{2x(x+2)}$$

$$= \lim_{x\to 0} \frac{x}{2x(x+2)} = \lim_{x\to 0} \frac{1}{2(x+2)} = \frac{1}{\lim_{x\to 0} 2(x+2)} = \frac{1}{2(0+2)} = \frac{1}{4} //$$

Example $\lim_{x\to 2} \frac{2-x}{3-\sqrt{x^2+5}}$

$$= \lim_{x\to 2} \frac{2-x}{3-\sqrt{x^2+5}} \cdot \frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}$$

$$= \lim_{x\to 2} \frac{(2-x)(3+\sqrt{x^2+5})}{3^2-(\sqrt{x^2+5})^2}$$

$$= \lim_{x\to 2} \frac{(2-x)(3+\sqrt{x^2+5})}{4-x^2} = \lim_{x\to 2} \frac{\cancel{(2-x)}(3+\sqrt{x^2+5})}{\cancel{(2-x)}(2+x)} = \frac{\lim_{x\to 2}(3+\sqrt{x^2+5})}{\lim_{x\to 2}(2+x)}$$

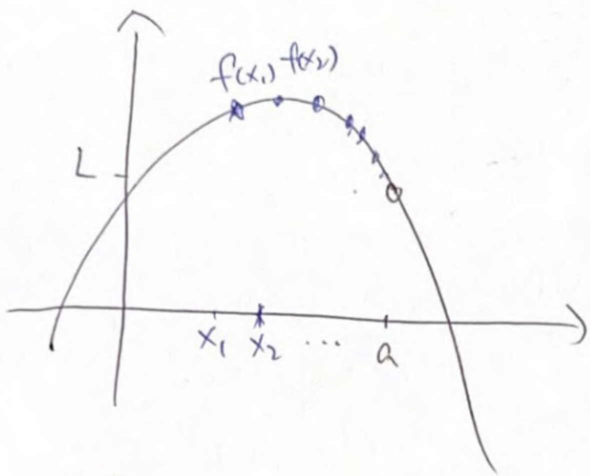
$$= \frac{6}{4} = \frac{3}{2} //$$

Thm (Sequential criterion for limit of functions)

Let $f: A \to B$ be a function, and $a, L \in \mathbb{R}$.

Then $\lim_{x \to a} f(x) = L$ if and only if

for any sequence $\{x_n\}$ with $x_n \neq a$ for any n satisfying $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = L$.



Consequence: If we can find two sequences $\{x_n\}, \{y_n\}$ (with $x_n \neq a$ and $y_n \neq a$ for any n) s.t.

$\lim_{n \to \infty} x_n = a$ and $\lim_{n \to \infty} y_n = a$ but

$\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$, then $\lim_{x \to a} f(x)$ does not exist.

Example Prove that $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

Solution: Let $\{x_n\} = \left\{\frac{1}{n\pi}\right\} = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$

Then $\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{n\pi} = 0$ and

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \sin \frac{1}{\frac{1}{n\pi}}$$
$$= \lim_{n \to \infty} \sin(n\pi)$$
$$= \lim_{n \to \infty} 0 = 0$$

Note: we use radian in calculus:
- π rad $= 180^\circ$
- $\frac{\pi}{2}$ rad $= 90^\circ$
- 2π rad $= 360^\circ$
- θ rad $= \left(\frac{180 \cdot \theta}{\pi}\right)^\circ$

Let $\{y_n\} = \left\{ \frac{1}{2n\pi + \frac{\pi}{2}} \right\} = \frac{1}{2\pi + \frac{\pi}{2}}, \frac{1}{4\pi+\frac{\pi}{2}}, \dots$

Then $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{1}{2n\pi + \frac{\pi}{2}} = 0$ and

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \sin \frac{1}{\frac{1}{2n\pi+\frac{\pi}{2}}}$$

$$= \lim_{n\to\infty} \sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$= \lim_{n\to\infty} 1 = 1$$

$\therefore \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$

$\therefore$ By sequential criterion, $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist $/\!/$

Thm (Squeeze thm for functions)

Let f, g, h be functions.

Suppose $f(x) \le g(x) \le h(x)$ for any $x \neq a$ on a neighborhood of a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$.

Then the limit of $g(x)$ at $x=a$ exists and we have

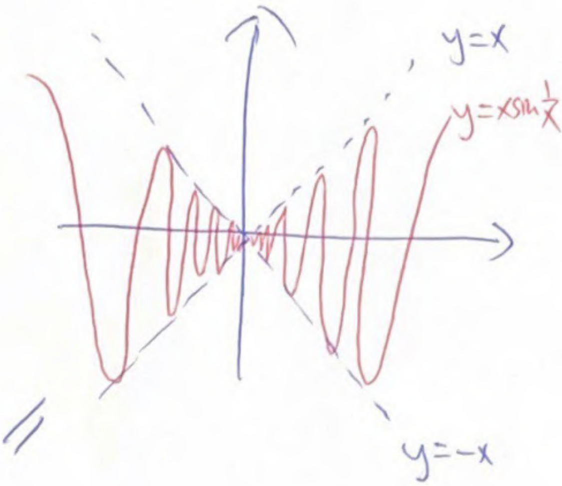
$$\lim_{x\to a} g(x) = L.$$

Thm If $f(x)$ is bounded (i.e. there exists M s.t. $|f(x)| < M$ for all x)

and $\lim_{x\to a} g(x) = 0$,

then $\lim_{x\to a} f(x) g(x) = 0$.

Example $\lim_{x \to 0} x \sin\frac{1}{x} = ?$



Solution Note that $\left|\sin\frac{1}{x}\right| \le 1$

and $\lim_{x \to 0} x = 0$

By squeeze thm, $\lim_{x \to 0} x \sin\frac{1}{x} = 0$ //

Recall: The Euler's number

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Def (Exponential function)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Prop ① $e^{x+y} = e^x e^y$ for any $x, y \in \mathbb{R}$.

② $e^x > 0$ for any $x \in \mathbb{R}$

③ e^x is strictly increasing.

Prop $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

Proof For any $-1 < x < 1$ with $x \neq 0$,

$$\frac{e^x - 1}{x} = \frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots$$

$$\leq 1 + \frac{x}{2} + \left(\frac{x^2}{3!} + \frac{x^2}{4!} + \cdots \right) \quad \left(\because \text{ for } -1<x<1, \; \begin{matrix} x^2 \geq x^3 \\ x^2 \geq x^4 \\ \vdots \end{matrix} \right)$$

$$\leq 1 + \frac{x}{2} + \left(\frac{x^2}{2\cdot 2} + \frac{x^2}{2\cdot 2\cdot 2} + \cdots \right) \quad \left(\because \begin{matrix} 3\cdot 2 \geq 2\cdot 2 \\ 4\cdot 3\cdot 2 \geq 2\cdot 2\cdot 2 \\ \vdots \end{matrix} \right)$$

$$= 1 + \frac{x}{2} + \left(\frac{x^2}{4} + \frac{x^2}{8} + \frac{x^2}{16} + \cdots \right)$$

$$= 1 + \frac{x}{2} + \frac{x^2}{2} \quad \left(\because \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{2} \right)$$

Also, $\displaystyle \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$

$$\geq 1 + \frac{x}{2} - \left(\frac{x^2}{4} + \frac{x^2}{8} + \frac{x^2}{16} + \cdots \right) = 1 + \frac{x}{2} - \frac{x^2}{2}$$

$$\therefore \quad 1 + \frac{x}{2} - \frac{x^2}{2} \leq \frac{e^x - 1}{x} \leq 1 + \frac{x}{2} + \frac{x^2}{2}$$

Now, note that

$$\lim_{x \to 0} \left(1 + \frac{x}{2} - \frac{x^2}{2} \right) = 1 \quad \text{and} \quad \lim_{x \to 0} \left(1 + \frac{x}{2} + \frac{x^2}{2} \right) = 1$$

By squeeze thm, $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x} = 1$

//

Example $\displaystyle \lim_{x \to 0} \frac{e^{5x} - e^{3x}}{x}$

$$= \lim_{x \to 0} \frac{e^{3x}(e^{2x} - 1)}{x}$$

$$= \left(\lim_{x \to 0} e^{3x} \right) \left(\lim_{x \to 0} \frac{e^{2x} - 1}{2x} \cdot 2 \right)$$

$$= 1 \cdot 1 \cdot 2$$

$$= 2$$

//

Def (Logarithmic function)

Notation: ($\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$)

$\ln : \mathbb{R}^+ \to \mathbb{R}$ with

$$y = \ln x \quad \text{if} \quad e^y = x.$$

In other words, $\ln x$ is the inverse function of e^x.

Prop ① $\ln(xy) = \ln x + \ln y$

② $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

③ $\ln(x^n) = n \ln x$ for any integer n.

Prop $\displaystyle\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$

Proof Let $y = \ln(x+1)$.

Then $e^y = 1 + x \iff x = e^y - 1$.

Also, as $y \to 0$, we have $x \to 0$.

$\therefore \displaystyle\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{y \to 0} \frac{y}{e^y - 1} = 1$ //

Infinite series of cosine and sine functions

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Prop $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Proof $$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$$

For any $-1<x<1$ with $x \neq 0$, we have

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!}\right) - \left(\frac{x^6}{7!} - \frac{x^8}{9!}\right) - \cdots \\ &\leq 1 - 0 - 0 - \cdots \\ &= 1 \end{aligned}$$

For $-1<x<1$,
$x^2 \geqslant x^4$
also,
$3! \leq 5!$
$\Leftrightarrow \frac{1}{3!} \geq \frac{1}{5!}$
$\therefore \frac{x^2}{3!} - \frac{x^4}{5!} \geq 0$

Also, $$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{6} + \left(\frac{x^4}{5!} - \frac{x^6}{7!}\right) + \left(\frac{x^8}{9!} - \frac{x^{10}}{11!}\right) + \cdots \\ &\geq 1 - \frac{x^2}{6} + 0 + 0 + \cdots \\ &= 1 - \frac{x^2}{6} \end{aligned}$$

$$\therefore \quad 1 - \frac{x^2}{6} \leq \frac{\sin x}{x} \leq 1$$

Note that $\lim_{x\to 0}\left(1 - \frac{x^2}{6}\right) = 1 = \lim_{x\to 0} 1$

By squeeze thm, $\lim_{x\to 0} \frac{\sin x}{x} = 1$ //