MATH 1010 F Lecture 4   14/9/2023

Last time

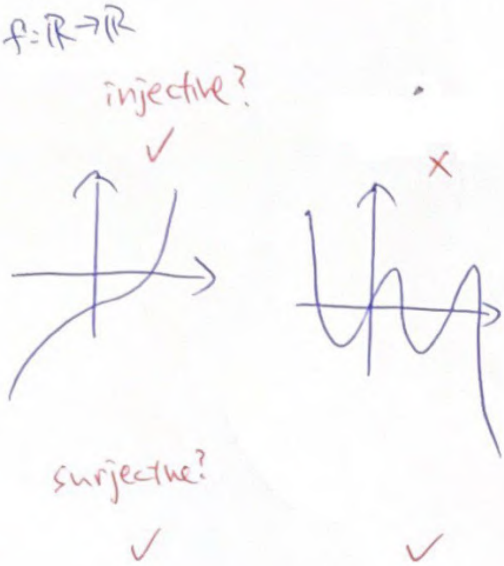
- The Euler's number

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.71828\ldots$$

- Squeeze theorem:

  If $a_n \le b_n \le c_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$,

  then $\lim_{n\to\infty} b_n = L$



- Functions $f: A \to B$
  - injective: (also known as "1-1", "one-to-one")

    For any $x_1, x_2 \in A$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$
  - surjective: (also known as "onto")

    For any $y \in B$, there exists $x \in A$ s.t. $y = f(x)$
  - bijective:

    both injective and surjective

Def (Inverse function)

Let $f: A \to B$ be a bijective function.

The inverse function of f is a function $f^{-1}: B \to A$ that "undoes the operation of f".

In other words, for any $y \in B$,

$$f^{-1}(y) = \text{the unique element } x \in A \text{ s.t. } f(x) = y.$$

Example

- $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x+1$:

$$y = x+1$$
$$x = y-1$$

we have $f^{-1}(y) = y-1$

- $f: [0,\infty) \to [0,\infty)$ with $f(x) = x^2$:

$$y = x^2$$
$$x = \sqrt{y}$$

we have $f^{-1}(y) = \sqrt{y}$

Prop For any bijective function $f: A \to B$, we have

- $f^{-1}(f(x)) = x$ for all $x \in A$
- $f(f^{-1}(y)) = y$ for all $y \in B$

Def (odd, even functions)

Let $f: A \to B$ be a function. We say that

① f is even if $f(-x) = f(x)$ for any x.

② f is odd if $f(-x) = -f(x)$ for any x.

Example

- $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^4 + x^2 + 1$ is even

$(\because f(-x) = (-x)^4 + (-x)^2 + 1 = x^4 + x^2 + 1 = f(x))$

- $f: \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$ with $f(x) = x - \frac{1}{x}$ is odd

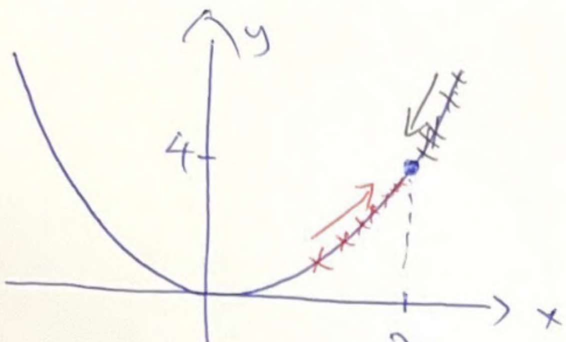
$(\because f(-x) = -x - \frac{1}{-x} = -x + \frac{1}{x} = -(x - \frac{1}{x}) = -f(x))$

- $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3 - 2x^2$: <u>not</u> even, <u>not</u> odd

$$\left(\because \ f(-x) = (-x)^3 - 2(-x)^2 = -x^3 - 2x^2 \right.$$
$$\left. f(-x) \neq f(x), \quad f(-x) \neq -f(x) \right)$$

Limit of functions

<u>Example</u> $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$



What happens when x is very close to 2?

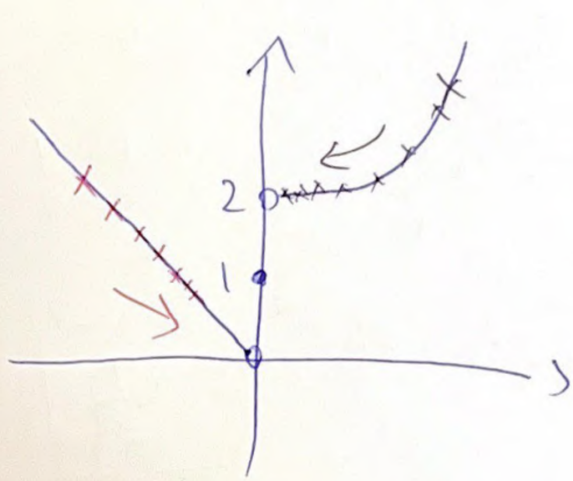
① As $x \to 2$ from the left (i.e. $x < 2$)

we have $f(x) \to 4$

② As $x \to 2$ from the right (i.e. $x > 2$)

we have $f(x) \to 4$

<u>Example</u> $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x^2 + 2 & \text{if } x > 0 \end{cases}$



What happens when x is very close to 0?

① As $x \to 0$ from the left

we have $f(x) \to 0$

different!

② As $x \to 0$ from the right

we have $f(x) \to 2$

Def (Limit of functions) (*informal*)

Let $f: A \to B$ be a function, and $a, L \in \mathbb{R}$.

We say that

① $\lim_{x \to a^-} f(x) = L$ (left-hand limit)

if $f(x)$ is close enough to L whenever

x is close enough to a and $\underline{x < a}$.

② $\lim_{x \to a^+} f(x) = L$ (right-hand limit)

if $f(x)$ is close enough to L whenever

x is close enough to a and $\underline{x > a}$.

③ $\lim_{x \to a} f(x) = L$ (two-sided limit)

if $f(x)$ is close enough to L whenever

x is close enough to a and $\underline{x \neq a}$

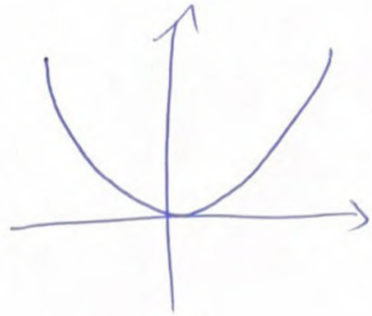
Remark
- The two-sided limit exists if and only if both the left-hand limit and right-hand limit exist and are equal, i.e.

$$\boxed{\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L}$$

- Whether f is defined at a or the value of f at a is not important for $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a} f(x)$!

Example

- $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$

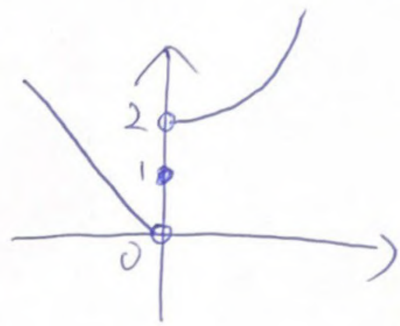


$$\lim_{x \to 2^-} f(x) = 4$$

$$\lim_{x \to 2^+} f(x) = 4$$

$$\therefore \lim_{x \to 2} f(x) = 4$$

- $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x^2 + 2 & \text{if } x > 0 \end{cases}$

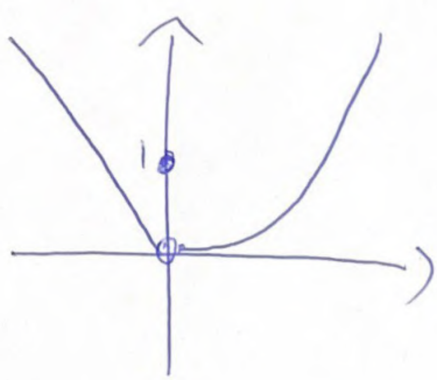


$$\lim_{x \to 0^-} f(x) = 0$$

$$\lim_{x \to 0^+} f(x) = 2$$

$\lim_{x \to 0} f(x)$ does not exist.

- $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x^2 & \text{if } x > 0 \end{cases}$



$$\lim_{x \to 0^-} f(x) = 0$$

$$\lim_{x \to 0^+} f(x) = 0$$

$$\therefore \lim_{x \to 0} f(x) = 0$$

the exact value of $f(0)$ doesn't matter!