MATH 1010F Lecture 3

12/9/2023

Last time

- Bounded sequence

  e.g. $a_n = \frac{1}{\sqrt{n}}$

  $1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots$ $\quad |a_n| < 2$ for all n

- If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded

- Monotone convergence theorem:

  If $\{a_n\}$ is bounded and monotonic, then $\{a_n\}$ is convergent.

Def (Series)

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

Example $\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} 1 = 1+1+\ldots+1 = n$$

Def (Convergence of infinite series)

We say that an infinite series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$$

is convergent if the sequence of partial sums $\{S_n\}$ where

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$ is convergent.

If $\sum_{k=1}^{\infty} a_k$ is convergent, we have

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k .$$

Example $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

The number e

Consider the sequence $\{S_n\}$ where

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \quad \left(\begin{array}{l} n! \\ = n\cdot(n-1)\cdot(n-2)\cdot \cdots \cdot(3)(2)(1) \end{array} \right)$$

Note that:

① $\{S_n\}$ is monotonic:

Note that we have

$$S_{n+1} - S_n = \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}\right) - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right)$$

$$= \frac{1}{(n+1)!} \geq 0 \qquad \text{for all } n$$

and hence $\{S_n\}$ is monotonic increasing.

② $\{S_n\}$ is bounded:

Note that we have $S_n \geq 0$ for all n.

Also,

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$\leq 1 + 1 + 1 = 3$$

$\therefore$ $0 \leq S_n \leq 3$ for all n

and hence $\{S_n\}$ is bounded.

By MCT, $\{S_n\}$ is convergent, i.e. $\lim_{n\to\infty} S_n$ exists.

The limit $e = \lim_{n\to\infty} S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

$\approx 2.71828\ldots$

is called the Euler's number.

Remark An alternative expression of the number e is

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828\ldots$$

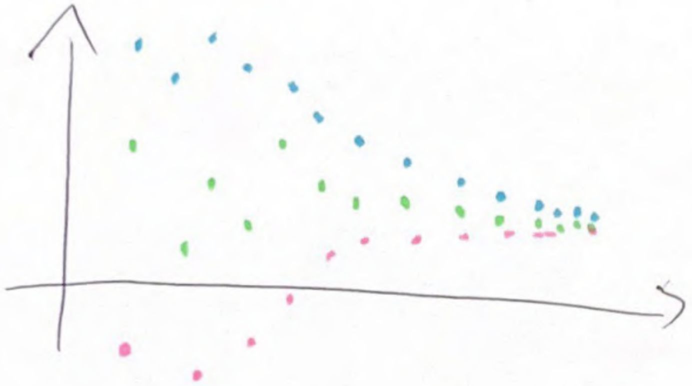
Thm (Squeeze theorem) (also known as the Sandwich thm)

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences.

If $a_n \le b_n \le c_n$ for all n

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$,

then $\{b_n\}$ is convergent and $\lim_{n\to\infty} b_n = L$.



Example $\lim_{n\to\infty} \frac{n+(-1)^n}{n+1} = ?$

Solution: Note that

$$\frac{n-1}{n+1} \le \frac{n+(-1)^n}{n+1} \le \frac{n+1}{n+1} = 1$$

Since $\lim_{n\to\infty}\frac{n-1}{n+1} = \lim_{n\to\infty}\frac{1-\frac{1}{n}}{1+\frac{1}{n}} = \frac{1-0}{1+0} = 1$ and $\lim_{n\to\infty} 1 = 1$,

by squeeze thm,

$$\lim_{n\to\infty}\frac{n+(-1)^n}{n+1} = 1$$

//

Thm If $\{a_n\}$ is bounded and $\lim_{n\to\infty} b_n = 0$,

then $\lim_{n\to\infty} a_n b_n = 0$

Proof $\because$ $\{a_n\}$ is bounded, there exists M s.t.

$$-M < a_n < M \quad \text{for all } n.$$

$$\therefore \; -M|b_n| < a_n b_n < M|b_n| \text{ for all } n.$$

Since $\lim_{n\to\infty} (-M|b_n|) = -M \lim_{n\to\infty} |b_n| = 0$

and $\lim_{n\to\infty} (M|b_n|) = M \lim_{n\to\infty} |b_n| = 0$,

by squeeze thm we have

$$\lim_{n\to\infty} a_n b_n = 0 \quad //$$

Example $\lim_{n\to\infty} \frac{\sin(n^2)}{2n+1} = ?$

Solution: Note that $-1 \le \sin(n^2) \le 1$ for all n

and $\lim_{n\to\infty} \frac{1}{2n+1} = 0$,

by thm $\Rightarrow \lim_{n\to\infty} \frac{\sin(n^2)}{2n+1} = 0 \quad //$

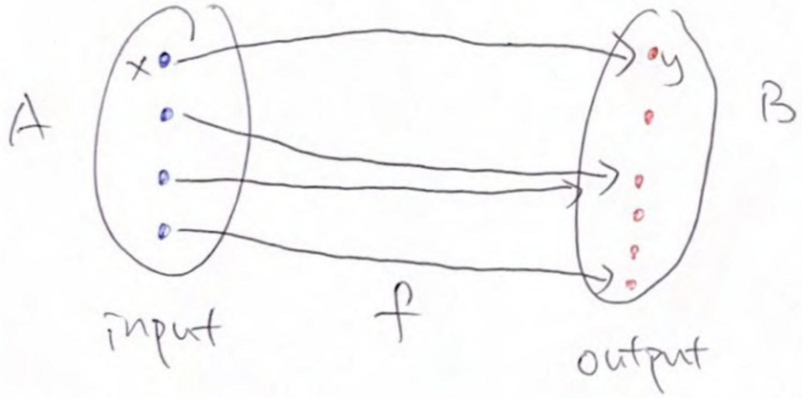
Functions

Def (function)

Let A, B be sets. A function $f: A \to B$ is a rule of assigning to each element of A an element of B.

A is called the domain of f.

B is called the codomain of f.



$f(x) = y$

We say that the image of x (under f) is y,

and x is a pre-image of y.

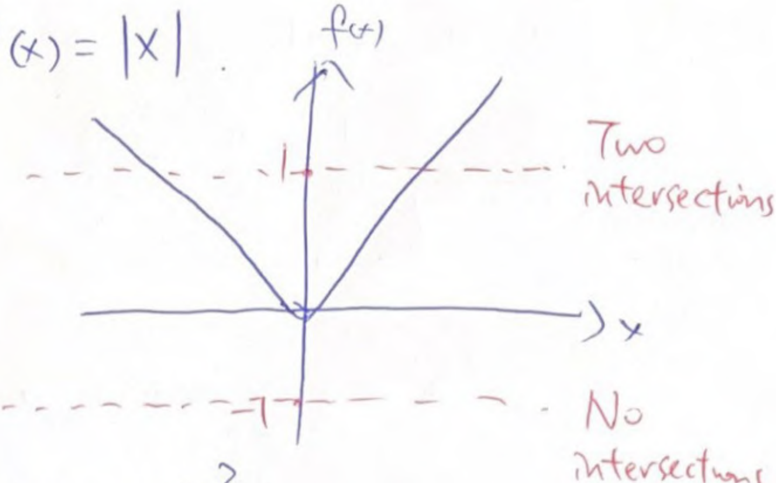
"is an element of"

Range of f $= f(A) = \{ f(x) \in B : x \in A \}$

$= \{ z \in B : z = f(x) \text{ for some } x \in A \}$

Example $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$.

domain codomain



range of $f = f(\mathbb{R})$

$= \{ z \in \mathbb{R} : z = |x| \text{ for some } x \in \mathbb{R} \}$

$= \{ z \in \mathbb{R} : z \geq 0 \}$

Def (intervals)

Let $a, b \in \mathbb{R}$.

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$ (open interval)

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval)

$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$

$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

$(a, \infty) = \{x \in \mathbb{R} : x > a\}$

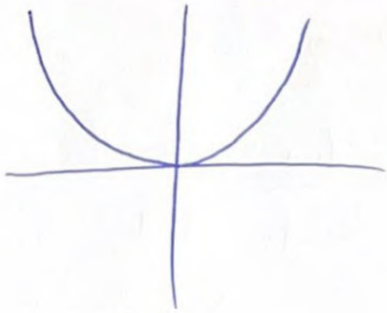
$(-\infty, \infty) = \mathbb{R}$

Example $2 \in [2,4]$ ($\because$ end points are included)

but $2 \notin (2,4)$

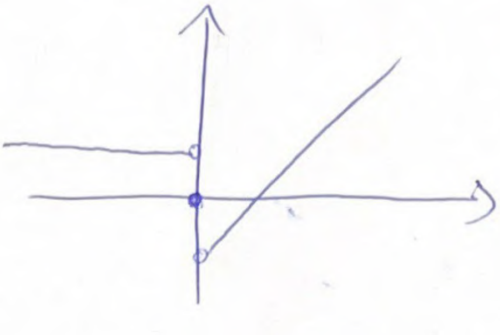
More examples of functions

- $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$



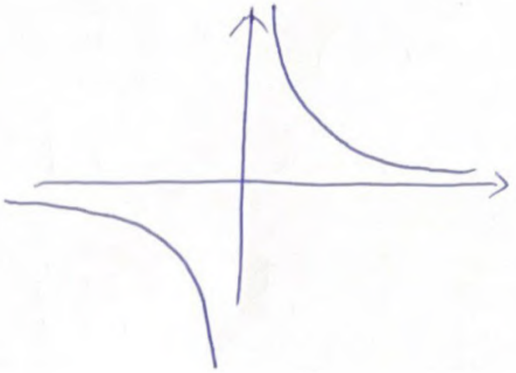
- $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x-1 & \text{if } x > 0 \end{cases}$$



- $f : \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$ with

$$f(x) = \frac{1}{x}$$



- $f: (-4, \infty) \to \mathbb{R}$ with

  $f(x) = \frac{1}{\sqrt{x+4}}$

- $f: [-2, 2] \to \mathbb{R}$ with

  $f(x) = \sqrt{4-x^2}$

Def (injective, surjective, bijective functions)

Let $f: A \to B$ be a function. f is said to be

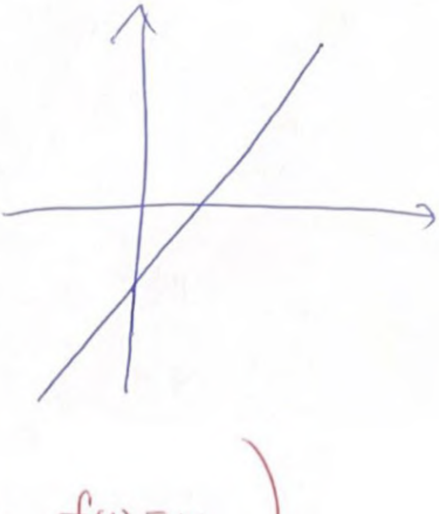
① injective if for any $x_1, x_2 \in A$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$

(or equivalently, whenever $f(x_1) = f(x_2)$ we have $x_1 = x_2$)

② surjective if for any $y \in B$, there exists $x \in A$ such that $f(x) = y$.

③ bijective if f is both injective and surjective.

Example $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x-3$



injective? ✓

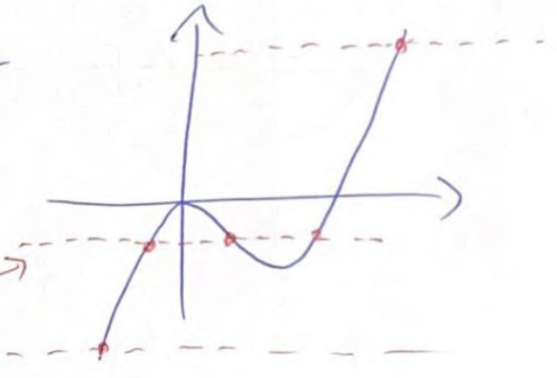
(if $2x_1-3 = 2x_2-3$, we must have $x_1 = x_2$)

surjective? ✓

(for any $y \in \mathbb{R}$, we can find $x = \frac{y+3}{2} \in \mathbb{R}$ s.t. $f(x) = y$)

bijective? ✓ ($\because$ both injective and surjective)

Example $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3 - 2x^2$



injective? X

(multiple intersections, so $f(x_1) = f(x_2)$ may not imply $x_1 = x_2$.)

surjective? ✓ (for any horizontal line, there is always an intersection, i.e. for any $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ s.t. $y = f(x)$)

bijective? X ($\because$ not injective)

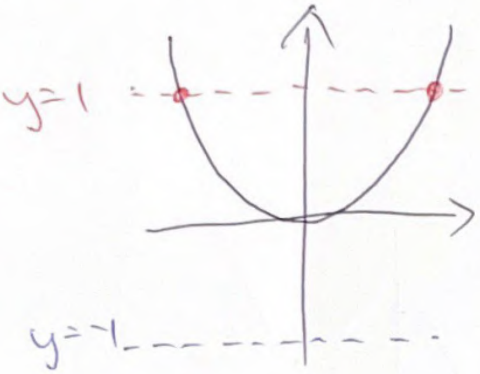
Remark The injectivity, surjectivity and bijectivity of a function also depend on the domain and codomain!

e.g.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$g: [0, \infty) \to \mathbb{R}$, $g(x) = x^2$

$h: [0, \infty) \to [0, \infty)$, $h(x) = x^2$



Injective?	X	✓	✓
Surjective?	X	X	✓
Bijective?	X	X	✓