MATH 1010F Lecture 2

7/9/2023

Last time

- Sequence
- Limit of sequence
- e.g. $a_n = \dfrac{(-1)^n}{n}$

  $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$

$\lim\limits_{n\to\infty} a_n = 0$

- e.g. $a_n = n^2$ $\quad 1, 4, 9, 16, \ldots$ $\quad \lim\limits_{n\to\infty} a_n = \infty$
- e.g. $a_n = (-1)^n$ $\quad -1, 1, -1, 1, \ldots$ $\quad \lim\limits_{n\to\infty} a_n$ does not exist.
- Properties of limit $(+, -, \times, \div)$
- <u>Monotonic sequence</u>:
  - monotonic increasing : $a_m \le a_n$ for any $m < n$

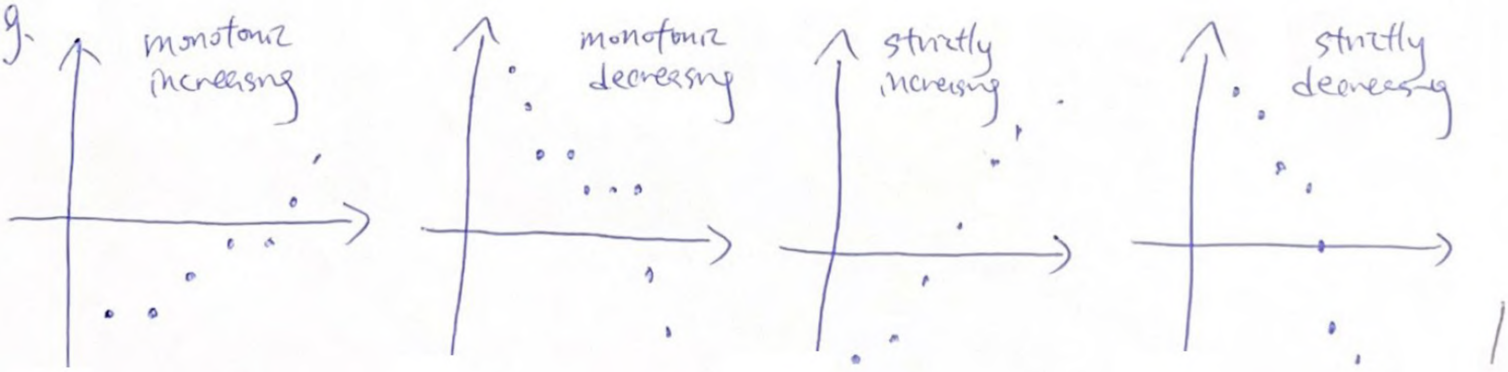
    (or equivalently, $a_n \le a_{n+1}$ for all n)
  - monotonic decreasing : $a_m \ge a_n$ for any $m < n$

    (or equivalently, $a_n \ge a_{n+1}$ for all n)
  - monotonic : either monotonic increasing or decreasing
  - strictly increasing : $a_m < a_n$ for any $m < n$

    (or equivalently, $a_n < a_{n+1}$ for all n)
  - Strictly decreasing : $a_m > a_n$ for any $m < n$

    (or equivalently, $a_n > a_{n+1}$ for all n)

e.g.



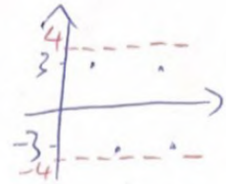
Def (Bounded sequence)

We say that $\{a_n\}$ is bounded if there exists a real number M such that

$$|a_n| < M \text{ for any } n \in \mathbb{Z}^+.$$

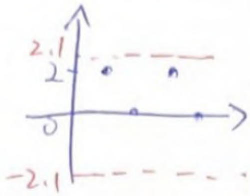
Example

- $3, -3, 3, -3, \ldots$



bounded ($\because |a_n| < 4$ for any n)

- $a_n = 1 - (-1)^n$: $2, 0, 2, 0, \ldots$

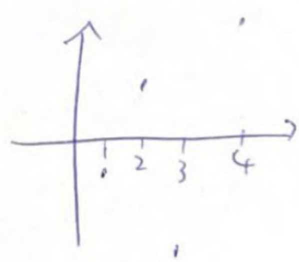


bounded ($\because |a_n| < 2.1$ for any n)

- $a_n = \frac{1}{n^2}$: $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

bounded ($\because |a_n| < 1.1$ for any n)

- $a_n = (-1)^n n$: $-1, 2, -3, 4, -5, 6, \ldots$



unbounded.

- $a_n = n^2$: $1, 4, 9, 16, \ldots$ unbounded

Thm If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded.

$$\boxed{\text{Convergent} \Rightarrow \text{Bounded}}$$

Remark: The converse is not true, i.e. bounded $\not\Rightarrow$ convergent

e.g. $a_n = 0, 1, 0, 1, \ldots$ is bounded but divergent.

Thm (Monotone convergence theorem)

If $\{a_n\}$ is <u>bounded</u> and <u>monotonic</u>, then $\{a_n\}$ is <u>convergent</u>

$$\boxed{\text{Bounded and Monotonic} \Rightarrow \text{Convergent}}$$

<u>Remark</u>: The theorem does <u>not</u> tell us what the limit is!

<u>Example</u> Let $a_1 = 1$, $a_n = \sqrt{1 + a_{n-1}}$ for $n \geq 2$.

① Show that $\{a_n\}$ is convergent

② Find its limit.

<u>Solution</u> ① (Note: $a_1 = 1$, $a_2 = \sqrt{1+1} = \sqrt{2}$, $a_3 = \sqrt{1+\sqrt{2}}$, … seem increasing!)

<u>Step 1</u> We first show that $\{a_n\}$ is monotonic increasing by induction, i.e. show that $a_{n+1} \geq a_n$ for all n.

- For $n = 1$, $a_2 = \sqrt{2} \geq 1 = a_1$
- Assume that we have $a_{k+1} \geq a_k$.

Now,
$$\begin{aligned} a_{k+2} &= \sqrt{1 + a_{k+1}} && \text{(by definition)} \\ &\geq \sqrt{1 + a_k} && \text{(by induction hypothesis)} \\ &= a_{k+1} && \text{(by def.)} \end{aligned}$$

$\therefore$ By induction, $\{a_n\}$ is <u>monotonic increasing</u>.

Step 2 We show that $1 \le a_n < 2$ for all n by induction

- For $n=1$, $1 \le a_1 = 1 < 2$.
- Assume that $1 \le a_k < 2$. Then

$$a_{k+1} = \sqrt{1+a_k} \ge \sqrt{1+1} > 1.$$

Also, $a_{k+1} = \sqrt{1+a_k} < \sqrt{1+2} = \sqrt{3} < 2$

$\therefore$ $1 \le a_n < 2$ for all n.

$\Rightarrow$ $\{a_n\}$ is bounded ($\because |a_n| < 2$ for all n)

Step 3 Since $\{a_n\}$ is monotonic and bounded, by monotone convergence thm, $\{a_n\}$ is convergent. //

(2) Let $\lim_{n\to\infty} a_n = L$. ← existence proved above.

$$a_n = \sqrt{1+a_{n-1}}$$

taking $\lim_{n\to\infty}$ on both sides,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{1+a_{n-1}} = \sqrt{1+\lim_{n\to\infty} a_{n-1}}$$

$$L = \sqrt{1+L}$$

$$\Rightarrow L^2 = 1+L$$

$$L^2 - L - 1 = 0$$

$$L = \frac{1\pm\sqrt{5}}{2}$$

Since $a_n > 0$ for all n, $L \ge 0$. Hence, $\frac{1-\sqrt{5}}{2}$ is rejected.

We have $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$ //

Next time:
- The number $e \approx 2.71828\ldots$
- Squeeze thm