MATH 1010F Lecture 1

5/9/2023

Def (Sequence of real numbers)

A sequence $\{a_n\}$ consists of $a_1, a_2, a_3, a_4, \ldots$

where each $a_i \in \mathbb{R}$.

is an element of / set of all real numbers

(Equivalently, a sequence is a function from $\mathbb{Z}^+$ to $\mathbb{R}$.)

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

set of all positive integers

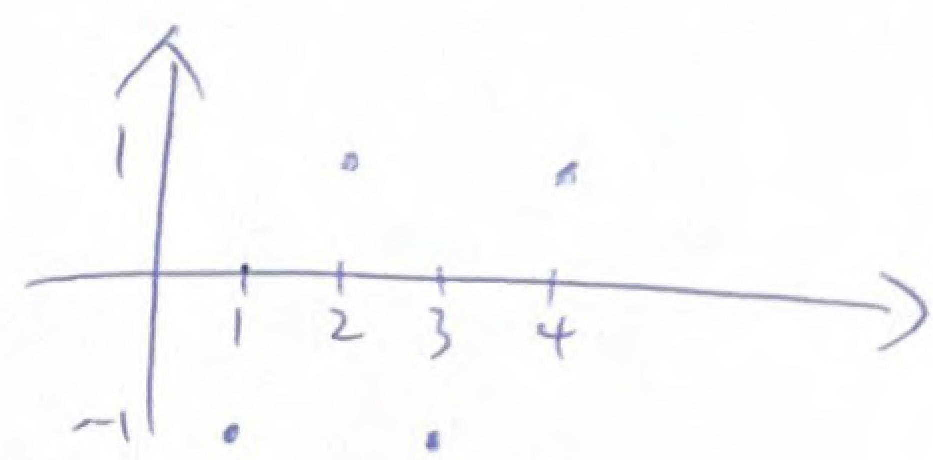
Example

$a_n = (-1)^n$

$a_1 = a_3 = a_5 = \cdots = -1$

$a_2 = a_4 = a_6 = \cdots = 1$

$-1, 1, -1, 1, -1, 1, \ldots$



Example (Arithmetic sequence)

$\{a_n\}$ such that $a_{n+1} - a_n = d$ for some constant d

$1, 3, 5, 7, 9, \ldots \qquad a_n = 2n - 1$

$19, 12, 5, -2, -9, \ldots \qquad a_n = 26 - 7n$

Example (Geometric sequence)

$\{a_n\}$ such that $a_{n+1} = r a_n$ for some constant r.

$1, 2, 4, 8, 16, \ldots \qquad a_n = 2^{n-1}$

$18, 6, 2, \frac{2}{3}, \frac{2}{9}, \ldots \qquad a_n = \frac{54}{3^n}$

<u>Example</u> (Fibonacci sequence)

$$\begin{cases} a_{n+2} = a_{n+1} + a_n \\ a_1 = a_2 = 1 \end{cases}$$

$1, 1, 2, 3, 5, 8, 13, \ldots$

<u>Example</u> $a_1 = 2, \quad a_n = \dfrac{1}{1 + a_{n-1}}$ for $n \geq 2$

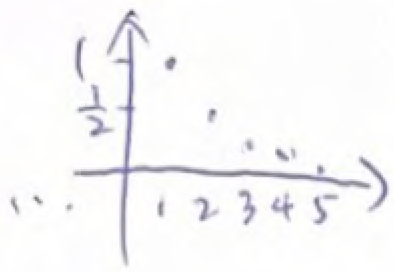
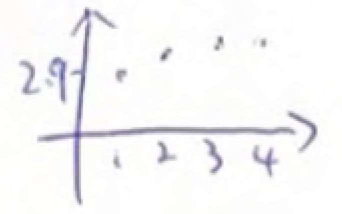
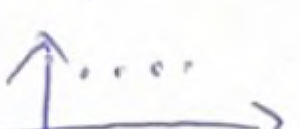
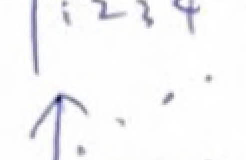
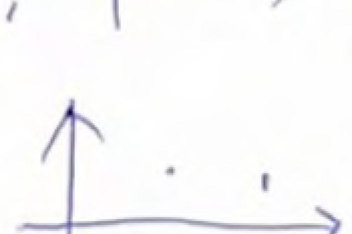
$\Rightarrow a_2 = \frac{1}{3}, \quad a_3 = \frac{3}{4}, \quad a_4 = \frac{4}{7}, \quad a_5 = \frac{7}{11}, \quad \ldots$

<u>Def</u> (Limit of sequence) $\boxed{\text{Informal}}$

- If a sequence $\{a_n\}$ approaches a real number L as n approaches infinity, then we say that a_n converges to L and write $\displaystyle\lim_{n\to\infty} a_n = L$.

- If no such L exists, then we say that $\displaystyle\lim_{n\to\infty} a_n$ does not exist and $\{a_n\}$ is said to be <u>divergent</u>.

<u>Example</u>

- $\{a_n\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$  $\displaystyle\lim_{n\to\infty} a_n = 0$ convergent.
- $\{a_n\} = 2.9, 2.99, 2.999, \ldots$  $\displaystyle\lim_{n\to\infty} a_n = 3$ convergent
- $\{a_n\} = 2, 2, 2, 2, 2, \ldots$  $\displaystyle\lim_{n\to\infty} a_n = 2$ convergent
- $\{a_n\} = 1, 3, 5, 7, 9, \ldots$  $\{a_n\}$ is divergent
- $\{a_n\} = -1, 1, -1, 1, \ldots$  divergent

Def (Limit of sequences) $\boxed{\text{Formal}}$

- Let $\{a_n\}$ be a sequence of real numbers.

Suppose there exists a real number L s.t. ← such that

for any $\varepsilon > 0$ (← epsilon), there exists a positive integer N s.t.

absolute value → $|a_n - L| < \varepsilon$ for any $n > N$.

Then we say that <u>a_n is convergent to L</u> and write

$$\underline{\lim_{n\to\infty} a_n = L}.$$

Otherwise, we say that a_n is divergent.

- Suppose for any $M > 0$, there exists a positive integer N s.t.

$$a_n > M \quad \text{for any } n > N.$$

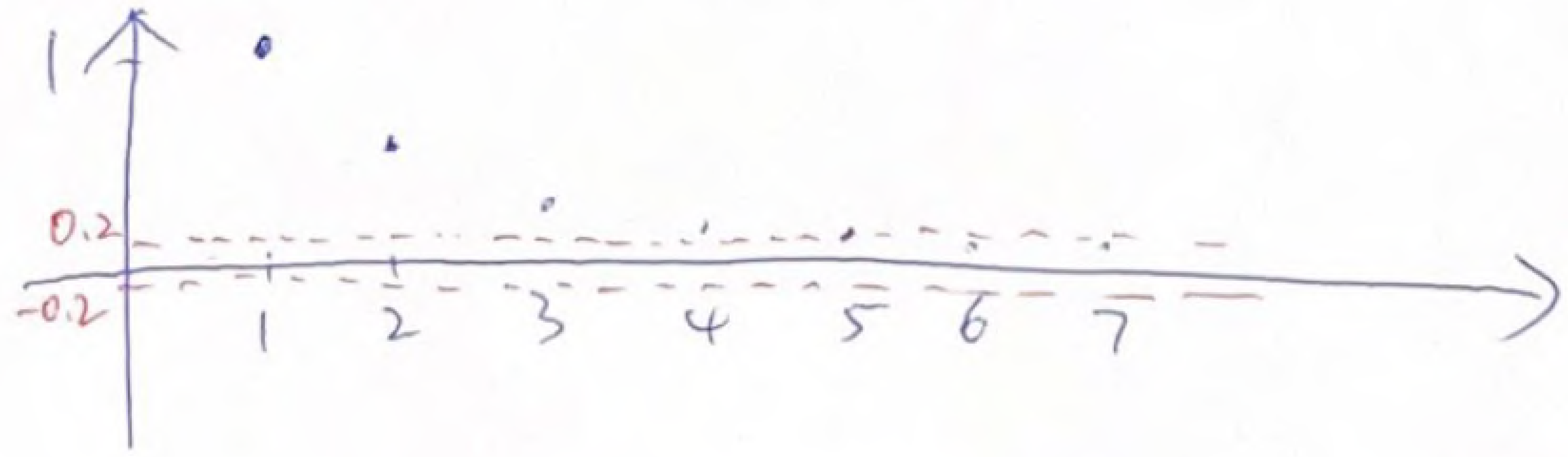
Then we say that <u>a_n tends to $+\infty$</u> as n tends to infinity

and write

$$\underline{\lim_{n\to\infty} a_n = +\infty}$$

(remark: it is still divergent)

Example $a_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$



$$\lim_{n\to\infty} a_n = 0.$$

Given $\varepsilon = 0.2$, we can take $N=5$ $\quad |a_n - 0| < 0.2$ for any $n > 5$

Given $\varepsilon = 0.01$, we can take $N = 100$ $\quad |a_n - 0| < 0.01$ for any $n > 100$

Properties of limit

Let $\{a_n\}$, $\{b_n\}$ be two sequences.

If $\lim_{n\to\infty} a_n = a$ is finite and $\lim_{n\to\infty} b_n = b$ is finite, then

1. $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$
2. $\lim_{n\to\infty} c a_n = ca$ for any real number c.
3. $\lim_{n\to\infty} a_n b_n = ab$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$.

Example

- $$\lim_{n\to\infty}\left(\frac{1}{n} - \frac{2}{n^2} + 1\right) = \lim_{n\to\infty}\frac{1}{n} - 2\lim_{n\to\infty}\frac{1}{n^2} + 1$$
$$= 0 - 0 + 1$$
$$= 1$$

Remark: The above properties may not hold if $\lim_{n\to\infty} a_n = \pm\infty$ or $\lim_{n\to\infty} b_n = \pm\infty$!

e.g. For $a_n = \frac{1}{n}$ and $b_n = n$, we have $\lim_{n\to\infty} a_n = 0$ but

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \frac{1}{n}\cdot n = \lim_{n\to\infty} 1 = 1 \neq 0.$$

e.g. $\lim_{n\to\infty} n$ diverges but $\lim_{n\to\infty} (n + (-n)) = \lim_{n\to\infty} 0 = 0$ converges

e.g. $$\lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0 \neq \frac{\lim_{n\to\infty} n}{\lim_{n\to\infty} n^2}$$

Remark For cases involving $\lim = \pm\infty$,

$$\begin{cases} \cdot\ \infty \pm L = \infty \\ \cdot\ -\infty \pm L = -\infty \\ \cdot\ \infty + \infty = \infty \\ \cdot\ -\infty - \infty = -\infty \\ \cdot\ L \cdot \infty = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases} \\ \cdot\ \dfrac{L}{\pm\infty} = 0 \end{cases}$$

For cases involving $\infty - \infty$, $\frac{\pm\infty}{\pm\infty}$, $\frac{0}{0}$, $0 \cdot \infty$: try further simplifying

Example
- $\displaystyle \lim_{n\to\infty} \frac{2n-5}{3n+1} = \lim_{n\to\infty} \frac{2-\frac{5}{n}}{3+\frac{1}{n}} = \frac{2-0}{3+0} = \frac{2}{3}$ //
- $\displaystyle \lim_{n\to\infty} \frac{2n^2+5}{n^3+1} = \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{5}{n^3}}{1+\frac{1}{n^3}} = \frac{0+0}{1+0} = 0$ //
- $\displaystyle \lim_{n\to\infty} \frac{n^2-1}{n+3} = \lim_{n\to\infty} \frac{n-\frac{1}{n}}{1+\frac{3}{n}} = \frac{\infty}{1} = \infty$ //
- $\displaystyle \lim_{n\to\infty} \left(n - \sqrt{n^2+4n}\right) = \lim_{n\to\infty} \left(n-\sqrt{n^2+4n}\right) \cdot \frac{n+\sqrt{n^2+4n}}{n+\sqrt{n^2+4n}}$

$$= \lim_{n\to\infty} \frac{n^2-(n^2+4n)}{n+\sqrt{n^2+4n}} \qquad \left(\because (a-b)(a+b) = a^2-b^2\right)$$

$$= \lim_{n\to\infty} \frac{-4n}{n+\sqrt{n^2+4n}}$$

$$= \lim_{n\to\infty} \frac{-4}{1+\sqrt{1+\frac{4}{n}}} = \frac{-4}{1+1} = -2 \quad //$$

How can we determine whether a sequence is convergent <u>without</u> directly finding the limit?

Def (Monotonic sequence)

- We say that $\{a_n\}$ is monotonic increasing (or monotonic decreasing) if for any $m < n$, we have $a_m \le a_n$ (or $a_m \ge a_n$).

  We say that $\{a_n\}$ is monotonic if it is either monotonic increasing or monotonic decreasing.
- We say that $\{a_n\}$ is strictly increasing (or strictly decreasing) if for any $m < n$, we have $a_m < a_n$ (or $a_m > a_n$).

Example

- $3, 3, 3, 3, \ldots$ monotonic
- $1, 1, 2, 2, 3, 3, \ldots$ monotonic increasing
- $10, 9, 8, 7, \ldots$ strictly decreasing
- $0, 1, 0, 1, \ldots$ not monotonic

next time:

(Monotone convergence theorem)

$$\boxed{\text{Monotonic and 'Bounded'} \Rightarrow \text{Convergent}}$$