A few words before we start

Pre-class notes for EPYMT TDG 2023

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1 Introduction

This is a really short introduction to let you know the difference between school math and how you should learn math. You often read books written like this:

Exercise 1.1 — Binomial theorem. I believe this is an easy exercise for you.

1. Show that $C_r^k + C_{r-1}^k = C_r^{k+1}$.

2. Let $n \geq 1$ be an integer, then

$$(x+y)^n = \sum_{r=0}^{n} C_r^n x^{n-r} y^r.$$

■

It is straight forward to solve part (2) by mathematical induction and use part (1) during the induction step. But how can we discover part (1) before we started doing part (2)? Here I suggest a way to “discover" part (1) inside a proof of part (2):

Solution. Let $S(n)$ be the statement $(x+y)^n = \sum_{r=0}^{n} C_r^n x^{n-r} y^r$. When $n=1$, left hand side is $x+y$ and is equal to the right hand side, so $S(1)$ is true.

Assume $S(k)$ is true for some positive integers k, consider $n=k+1$:

$$\begin{aligned}
L.H.S. &= (x+y)^{k+1} \\
&= (x+y)\sum_{r=0}^{k} C_r^k x^{k-r} y^r \\
&= \sum_{r=0}^{k} C_r^k x^{k+1-r} y^r + C_r^k x^{k-r} y^{r+1} \\
&= \sum_{r=1}^{k} C_r^k x^{k+1-r} y^r + x^{k+1} + \sum_{r=1}^{k+1} C_{r-1}^k x^{k-r+1} y^r
\end{aligned}$$

where the last step is there because we observe the R.H.S. we want:

$$R.H.S. = \sum_{r=0}^{k+1} C_r^{k+1} x^{k+1-r} y^r = \sum_{r=1}^{k+1} C_r^{k+1} x^{k+1-r} y^r + x^{k+1}$$

and we create the $x^{k+1-r}y^r$ term on the left hand side by re-indexing the summation. By comparing the coefficient of $x^{k+1-r}y^r$ on both sides, we ask whether

$$C_r^k + C_{r-1}^k = C_r^{k+1}$$

is true. ■

It is important to ask why a lemma exists. A well-written book prepares you with lots of lemmas and more-than-necessary information for proving a theorem, making the proof of the theorem you like rather easy or is a trivial consequence of the previous propositions. But sometimes those propositions are there because people attacked the theorem first.

Exercise 1.2 Suppose you know the exponential function can be written as

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Explain why you need the following proposition to prove $e^{x+y} = e^x e^y$:

$$\lim_{N\to\infty} \left| \sum_{p=0}^{2N} \frac{(x+y)^p}{p!} - \sum_{p=0}^{N} \frac{x^p}{p!} \sum_{p=0}^{N} \frac{y^p}{p!} \right| = 0.$$

(And try to prove $e^{x+y} = e^x e^y$ by yourself!) ▪

! Don't tell me you want to prove by calculator.