MMAT5010 2223 Assignment 8

Q1.(a) To show $M^\perp$ is closed, we need to show that for any sequence $\{x_n\}_{n=1}^\infty \subset M^\perp$ such that $x_n \to x$ for some $x \in X$, we have $x \in M^\perp$. In fact, for any $y \in M$,

$$(x, y) = (\lim_{n\to\infty} x_n, y) = \lim_{n\to\infty} (x_n, y) = 0.$$

Hence $x \in M^\perp$. Therefore, $M^\perp$ is closed.

(b) Firstly we show $(\overline{M})^\perp \subset M^\perp$. For any $x \in (\overline{M})^\perp$, any $y \in M \subset \overline{M}$, we have $(x, y) = 0$. Hence $x \in M^\perp$. Therefore $(\overline{M})^\perp \subset M^\perp$.

Now we show $M^\perp \subset (\overline{M})^\perp$. We need to show for any $x \in M^\perp$, any $y \in \overline{M}$, we have $(x, y) = 0$. In fact, since $y \in \overline{M}$, there exists a sequence $\{y_n\}_{n=1}^\infty \subset M$ such that $y_n \to y$. Then

$$(x, y) = (x, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} (x, y_n) = 0.$$

Hence $x \in (\overline{M})^\perp$. Therefore, $M^\perp \subset (\overline{M})^\perp$. In conclusion, $(\overline{M})^\perp = M^\perp$.

Q2. For any $x, y \in H_1$, we have $x = \sum_{n=1}^\infty (x, e_n)e_n$, $y = \sum_{n=1}^\infty (y, e_n)e_n$. Then

$$(x, y) = (\sum_{n=1}^\infty (x, e_n)e_n, \sum_{n=1}^\infty (y, e_n)e_n) = \sum_{n=1}^\infty (x, e_n)\overline{(y, e_n)}.$$

And

$$Ux = U(\sum_{n=1}^\infty (x, e_n)e_n) = \sum_{n=1}^\infty (x, e_n)Ue_n = \sum_{n=1}^\infty (x, e_n)f_n.$$

Similarly, $Uy = \sum_{n=1}^\infty (y, e_n)f_n$. Therefore,

$$(Ux, Uy) = (\sum_{n=1}^\infty (x, e_n)f_n, \sum_{n=1}^\infty (y, e_n)f_n) = \sum_{n=1}^\infty (x, e_n)\overline{(y, e_n)} = (x, y).$$