MMAT5010 2223 Assignment 7

Q1. Notice that

$$\begin{aligned}|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|.\end{aligned}$$

Since $x_n \to x$ and $y_n \to y$, we obtain $\{x_n\}$ is bounded and $\|x_n - x\| \to 0$, $\|y_n - y\| \to 0$. Hence $|\langle x_n, y_n\rangle - \langle x, y\rangle| \to 0$. That is, $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

Q2. To show $\langle \cdot, \cdot\rangle_{X\times Y}$ is an inner product, we need to show
(i) $\langle (x,y),(x,y)\rangle_{X\times Y} \geq 0$ for any $(x,y) \in X\times Y$,
(ii) $\langle (x,y),(x,y)\rangle_{X\times Y} = 0$ iff $x = 0_X, y = 0_Y$,
(iii) $\overline{\langle (x_1,y_1),(x_2,y_2)\rangle}_{X\times Y} = \langle (x_2,y_2),(x_1,y_1)\rangle_{X\times Y}$,
for any $(x_1,y_1),(x_2,y_2) \in X\times Y$,
(iv) $\langle \alpha(x_1,y_1) + \beta(x_2,y_2), (x_3,y_3)\rangle_{X\times Y} = \alpha\langle (x_1,y_1),(x_3,y_3)\rangle_{X\times Y} + \beta\langle (x_2,y_2),(x_3,y_3)\rangle_{X\times Y}$ for any $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in X\times Y$ and $\alpha, \beta \in \mathbb{C}$.

For (i), $\langle (x,y),(x,y)\rangle_{X\times Y} = \langle x,x\rangle_X + \langle y,y\rangle_Y \geq 0$.
For (ii), $\langle (x,y),(x,y)\rangle_{X\times Y} = 0$ iff $\langle x,x\rangle_X = 0$, $\langle y,y\rangle_Y = 0$ iff $x = 0_X, y = 0_Y$.
For (iii), $\overline{\langle (x_1,y_1),(x_2,y_2)\rangle}_{X\times Y} = \overline{\langle x_1,x_2\rangle_X + \langle y_1,y_2\rangle_Y} = \langle x_2,x_1\rangle_X + \langle y_2,y_1\rangle_Y = \langle (x_2,y_2),(x_1,y_1)\rangle_{X\times Y}$.
For (iv), $\langle \alpha(x_1,y_1)+\beta(x_2,y_2),(x_3,y_3)\rangle_{X\times Y} = \langle \alpha x_1+\beta x_2, x_3\rangle_X + \langle \alpha y_1+\beta y_2, y_3\rangle_Y$ $= \alpha\langle x_1,x_3\rangle_X + \beta\langle x_2,x_3\rangle_X + \alpha\langle y_1,y_3\rangle_Y + \beta\langle y_2,y_3\rangle_Y = \alpha\langle (x_1,y_1),(x_3,y_3)\rangle_{X\times Y} + \beta\langle (x_2,y_2),(x_3,y_3)\rangle_{X\times Y}$.

Now we prove $X\times Y$ is a Hilbert space. Let $\{(x_n,y_n)\}$ be a Cauchy sequence in $X\times Y$, we need to show that there eixst $x \in X, y \in Y$ such that $\|(x_n - x, y_n - y)\|_{X\times Y} \to 0$. In fact, $\|x_n - x_m\|_X = \sqrt{\langle x_n - x_m, x_n - x_m\rangle_X} \leq \sqrt{\langle (x_n - x_m, y_n - y_m),(x_n - x_m, y_n - y_m)\rangle_{X\times Y}} = \|(x_n - x_m, y_n - y_m)\|_{X\times Y} \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in X. Similarly, $\{y_n\}$ is a Cauchy sequence in Y. Since X, Y are both Hilbert spaces, there exist $x \in X, y \in Y$ such that $x_n \to x, y_n \to y$. Hence
$\|(x_n - x, y_n - y)\|_{X\times Y} = \sqrt{\langle (x_n - x, y_n - y),(x_n - x, y_n - y)\rangle_{X\times Y}}$
$= \sqrt{\langle x_n - x, x_n - x\rangle_X + \langle y_n - y, y_n - y\rangle_Y} \leq \|x_n - x\|_X + \|y_n - y\|_Y \to 0$.
Therefore $X\times Y$ is a Hilbert space.