MMAT5010 2223 Assignment 3

Q1. Because all norms are equivalent in finite dimensional spaces, it only needs to show A is a bounded linear operator from $(\mathbb{K}^n, \|\cdot\|_\infty)$ to itself, here $\|\cdot\|_\infty$ is the supremum norm.

For any $x \in \mathbb{K}^n$, write $x = (x_1, x_2, ..., x_n)$. Let $e_i = (0, ..., 0, 1, 0, ..., 0)$(the i-th entry is 1, others are 0). Then

$$\|Ax\|_\infty = \|A(\sum_{i=1}^{n} x_i e_i)\|_\infty \leq \sum_{i=1}^{n} |x_i| \|Ae_i\|_\infty \leq (n \max_{i=1,...,n} \|Ae_i\|_\infty)\|x\|_\infty.$$

Hence A is a bounded linear operator with respect to $\|\cdot\|_\infty$ and therefore is continuous with respect to any norm $\|\cdot\|$ defined on $\mathbb{K}^n$.

Q2. Let (x_n) be a sequence in A. Since A is compact with respect to the norm $\|\cdot\|_1$, there exists a subsequence (x_{n_k}) and $x \in A$ such that $\|x_{n_k} - x\|_1 \to 0$ as $k \to \infty$. Because $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, we have $\|x_{n_k} - x\|_2 \leq C\|x_{n_k} - x\|_1 \to 0$ as $k \to \infty$. Hence (x_{n_k}) also converges in A with respect to the norm $\|\cdot\|_2$. So A is also compact with respect to the norm $\|\cdot\|_2$.

Q3. Suppose (x_n) is a sequence in $\ell_1, x_n \to x \in \ell_1$ in $\|\cdot\|_1$-norm. Because $\|\cdot\|_\infty \leq \|\cdot\|_1$, we have $\|x_n - x\|_\infty \leq \|x_n - x\|_1 \to 0$ as $n \to \infty$. Then $x_n \to x$ in $\|\cdot\|_\infty$-norm.

The converse statement is: suppose (x_n) is a sequence in ℓ_1, $x_n \to x \in \ell_1$ in $\|\cdot\|_\infty$-norm, then $x_n \to x$ in $\|\cdot\|_1$-norm.

This statement is disproved by finding $(x_n), x, x_n \to x$ in $\|\cdot\|_\infty$ but $x_n \nrightarrow x$ in $\|\cdot\|_1$. Define

- $x_1 = (1, 0, 0, ...) \in \ell_1$
- $x_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, ...) \in \ell_1$
- $x_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, ...) \in \ell_1$
- ...

Then $\|x_n - 0\|_\infty = \frac{1}{n} \to 0$, but $\|x_n - 0\|_1 = 1$ for all n.