MMAT5010 2223 Assignment 1

Q1. Clearly $q(x) \geq 0$ for any $x \in X$. It remains to check

(i) $q(x) = 0$, if and only if $x = 0_X$,

(ii) $q(tx) = |t|q(x)$ for $t \in \mathbb{R}, x \in X$,

(iii) $q(x_1 + x_2) \leq q(x_1) + q(x_2)$ for $x_1, x_2 \in X$.

For (i), if $q(x) = 0$, then $\|x\|_X \leq \|x\|_X + \|Tx\|_Y = 0$, therefore $\|x\|_X = 0$, hence $x = 0_X$. Conversely, $q(0_X) = \|0_X\|_X + \|T(0_X)\|_Y = \|0_X\|_X + \|0 \cdot T(0_X)\|_Y = 0$.

For (ii), suppose $t \in \mathbb{R}, x \in X$, then $q(tx) = \|tx\|_X + \|T(tx)\|_Y = \|tx\|_X + \|tTx\|_Y = |t|(\|x\|_X + \|Tx\|_Y) = |t|q(x)$.

For (iii), let $x_1, x_2 \in X$, then

$$\begin{aligned} q(x_1 + x_2) &= \|x_1 + x_2\|_X + \|T(x_1 + x_2)\|_Y \\ &= \|x_1 + x_2\|_X + \|Tx_1 + Tx_2\|_Y \\ &\leq \|x_1\|_X + \|x_2\|_X + \|Tx_1\|_Y + \|Tx_2\|_Y \\ &= q(x_1) + q(x_2). \end{aligned}$$

Q2. ($\Rightarrow$) Suppose X is a Banach space, we want to show that S_X is complete. Let (x_n) be a Cauchy sequence in S_X. Because X is complete, (x_n) converges to some $x \in X$. Since $\|x\| = \|\lim_{n\to\infty} x_n\| = \lim_{n\to\infty}\|x_n\| = 1$, we have $x \in S_X$.

($\Leftarrow$) Suppose S_X is complete. Let (x_n) be a Cauchy sequence in X, since

$$|\|x_n\| - \|x_m\|| \leq \|x_n - x_m\|,$$

$(\|x_n\|)$ is a Cauchy sequence in $\mathbb{R}$. $\mathbb{R}$ is complete, $(\|x_n\|)$ converges to some $L \in \mathbb{R}$. If $L = 0$, then $\lim_{n\to\infty}\|x_n - 0\| = 0$, i.e. $x_n \to 0$. If $L \neq 0$, we may assume that $x_n \neq 0$, then $(x_n/\|x_n\|)$ is a Cauchy sequence in S_X. (**Fact:** if (λ_n) is a Cauchy sequence in $\mathbb{R}$, (x_n) a Cauchy sequence in X, then $(\lambda_n x_n)$ is a Cauchy sequence in X.) By assumption, $x_n/\|x_n\|$ is convergent to some $s \in S_X$, so $x_n = \|x_n\| \frac{x_n}{\|x_n\|}$ is convergent to $Ls \in X$.

Q3. (a) Clearly $q(x, y) \geq 0$ for any $(x, y) \in X \oplus Y$ and $q(0_X, 0_Y) = 0$. It remains to check

(i) If $q(x, y) = 0$, then $x = 0_X$ and $y = 0_Y$,

(ii) $q(tx, ty) = tq(x, y)$ for $t \in \mathbb{R}, x \in X, y \in Y$,

(iii) $q(x_1 + x_2, y_1 + y_2) \leq q(x_1, y_1) + q(x_2, y_2)$ for $x_1, x_2 \in X, y_1, y_2 \in Y$.

For (i), suppose $q(x, y) = 0$, then $\|x\|_X + \|y\|_Y = 0$, therefore $\|x\|_X = \|y\|_Y = 0$, hence $x = 0_X$ and $y = 0_Y$.

For (ii), suppose $t \in \mathbb{R}, x \in X, y \in Y$, then $q(tx, ty) = \|tx\|_X + \|ty\|_Y = t(\|x\|_X + \|y\|_Y) = tq(x, y)$.

For (iii), let $x_1, x_2 \in X, y_1, y_2 \in Y$, then

$$\begin{aligned} q(x_1 + x_2, y_1 + y_2) &= \|x_1 + x_2\|_X + \|y_1 + y_2\|_Y \\ &\leq \|x_1\|_X + \|x_2\|_X + \|y_1\|_Y + \|y_2\|_Y \\ &= q(x_1, y_1) + q(x_2, y_2). \end{aligned}$$

(b)($\Rightarrow$) Suppose that $X \oplus Y$ is a Banach space. By summetry it suffices to show that X is a Banach space. Let (x_n) be a Cauchy sequence in X, then $\|x_n\|_X = q(x_n, 0_Y)$, so $(x_n, 0_Y)$ is a Cauchy sequence in $X \oplus Y$.

By assumption, $\lim_{n\to\infty}(x_n, 0_Y) = (x, y)$ exists. Since $\|x_n - x\| = q(x_n - x, 0_Y) \leq q(x_n - x, 0_Y - y)$, it follows that (x_n) converges to x.

($\Leftarrow$) Suppose both X, Y are Banach spaces. Let (x_n, y_n) be a Cauchy sequence in $X \oplus Y$, note that $\|x_n\|_X \leq q(x_n, y_n)$, therefore, (x_n) is a Cauchy sequnce in X, by assumption (x_n) converges to some $x \in X$, i.e. $\lim_{n\to\infty}\|x - x_n\|_X = 0$. Similarly, (y_n) is convergent to some $y \in Y$, i.e. $\lim_{n\to\infty}\|y - y_n\|_Y = 0$. So $\lim_{n\to\infty} q(x_n - x, y_n - y) = 0$ and hence (x_n, y_n) converges to (x, y).