Math 6261 23-04-21

§ 5.3 Doob's Martingale convergence Thm

Def. Let $(\Omega, \mathcal{F}, P)$ be a prob. space, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$.

A sequence $X_1, X_2, \cdots,$ of r.v.'s with X_n being $\mathcal{F}_n$-measurable and $E|X_n| < \infty$, is said to

a submartingale if $E(X_{n+1} | \mathcal{F}_n) \geq X_n$ a.s

a martingale if $E(X_{n+1} | \mathcal{F}_n) = X_n$ a.s

a supermartingale if $E(X_{n+1} | \mathcal{F}_n) \leq X_n$ a.s.

The fundamental result about martingale is the following theorem of Doob:

Thm 5.6 (Submartingale convergence Thm)

Every L^1 bounded submartingale (i.e. $\sup_n E|X_n| < \infty$) converges a.s.

Below we prove this result by following the approach in the book "Ergodic theory" of K. Petersen.

we first give the following.

Lem 5.7 (Krickeberg Decomposition)

Every L^1-bounded submartingale $\{X_n\}$ is the difference of a non-negative martingale $\{M_n\}$ and a non-negative supermartingale $\{S_n\}$. That is, $X_n = M_n - S_n$.

Pf. Notice that X_n^+ is a non-negative submartingale.

Indeed,

$$E(X_{n+1}^+ \mid \mathcal{F}_n) \geq E(X_{n+1} \mid \mathcal{F}_n) = X_n.$$

Since $E(X_{n+1}^+ \mid \mathcal{F}_n) \geq 0$, we obtain $E(X_{n+1}^+ \mid \mathcal{F}_n) \geq X_n^+$, thus, $\{X_n^+\}$ is a non-negative submartingale.

Next fix m. Let $n \geq m$. Then

$$E(X_{n+1}^+ \mid \mathcal{F}_m) = E(E(X_{n+1}^+ \mid \mathcal{F}_n) \mid \mathcal{F}_m)$$
$$\geq E(X_n^+ \mid \mathcal{F}_m).$$

Hence $E(X_n^+ \mid \mathcal{F}_m)$ increases in n.

Write $M_m = \lim_{n\to\infty} E(X_n^+ | \mathcal{F}_m)$.

We claim that M_m is a martingale. Observe that

$$E(M_{n+1} | \mathcal{F}_n) = E\left(\lim_{k\to\infty} E(X_k^+ | \mathcal{F}_{n+1}) \,\middle|\, \mathcal{F}_n\right)$$

$$= \lim_{k\to\infty} E\left(E(X_k^+ | \mathcal{F}_{n+1}) \,\middle|\, \mathcal{F}_n\right)$$

$$= \lim_{k\to\infty} E(X_k^+ | \mathcal{F}_n)$$

$$= M_n.$$

Hence M_n is a Martingale.

Notice that $M_n \in L^1$, since $E(M_n) = \lim_{k\to\infty} E\left(E(X_k^+ | \mathcal{F}_n)\right)$

$$= \lim_{k\to\infty} E(X_k^+)$$

$$\leq \sup_k E|X_k| < \infty.$$

Finally, let $S_n = M_n - X_n$.

Recall that $M_n \geq E(X_{n+1}^+ | \mathcal{F}_n) \geq E(X_{n+1} | \mathcal{F}_n) \geq X_n$,

So $S_n \geq 0$. Moreover, $E(S_n) \leq E(M_n) + E(|X_n|)$

$$\leq 2 \sup_k E|X_k| < \infty.$$

Hence $S_n \in L^1$. Now

$$E(S_{n+1} \mid \mathcal{F}_n) = E(M_{n+1} \mid \mathcal{F}_n) - E(X_{n+1} \mid \mathcal{F}_n)$$

$$< M_n - X_n = S_n.$$

So S_n is a supermartingale. ▨

Lem 5.8 (optional sampling).

Let $\{X_n\}$ be a non-negative supermartingale w.r.t $\mathcal{F}_n$

Let σ, τ be stopping times, that is,

$$\{\omega : \sigma(\omega) \leq n\} \in \mathcal{F}_n \quad \text{and} \quad \{\omega ; \tau(\omega) \leq n\} \in \mathcal{F}_n$$

for $n = 1, 2, \cdots$. Define

$$X_\tau(\omega) = \begin{cases} X_{\tau(\omega)}(\omega) & \text{if} \quad \tau(\omega) < \infty \\ 0 & \text{otherwise}, \end{cases}$$

and similarly for $X_\sigma(\omega)$. Suppose that $\sigma \leq \tau$, then

$$E(X_\tau) \leq E(X_\sigma).$$

Pf. Fix $m \geq 1$. For $n \geq m$, we have

$$\int_{\{\sigma=m\}} X_{\tau\wedge n} dP = \int_{\{\sigma=m,\ \tau\leq n\}} X_\tau \, dP + \int_{\{\sigma=m,\ \tau>n\}} X_n \, dP$$

$$\geq \int_{\{\sigma=m,\ \tau\leq n\}} X_\tau \, dP + \int_{\{\sigma=m,\ \tau>n\}} X_{n+1} \, dP$$

(since $\{\sigma=m, \tau>n\} \in \mathcal{F}_n$,
$X_n \geq E(X_{n+1} | \mathcal{F}_n)$)

$$= \int_{\{\sigma=m,\ \tau\leq n\}} X_{\tau\wedge(n+1)} \, dP + \int_{\{\sigma=m,\ \tau>n\}} X_{\tau\wedge(n+1)} dP$$

$$= \int_{\{\sigma=m\}} X_{\tau\wedge(n+1)} \, dP. \qquad (1)$$

It follows that for $n \geq m$,

$$\int_{\{\sigma=m\}} X_\sigma \, dP = \int_{\{\sigma=m\}} X_m \, dP = \int_{\{\sigma=m\}} X_{\tau\wedge m} \, dP$$

$$\geq \int_{\{\sigma=m\}} X_{\tau\wedge n} \, dP \quad \text{(by (1))}.$$

$$\geq \int_{\{\sigma=m,\ \tau<\infty\}} X_{\tau\wedge n} \, dP$$

Thus $$\int_{\{\sigma=m\}} X_\sigma \, dP \geq \int_{\{\sigma=m,\ \tau<\infty\}} \varliminf_{n\to\infty} X_{\tau\wedge n} \, dP$$

$$= \int_{\{\sigma=m,\ \tau<\infty\}} X_\tau \, dP$$

$$= \int_{\{\sigma=m\}} X_\tau \, dP. \quad \left(\text{since } X_\tau = 0 \text{ on } \{\tau=+\infty\}\right)$$

Summing on m gives

$$E(X_\sigma) \geq E(X_\tau). \quad \boxed{\checkmark}$$

Pf of Submartingale convergence Thm :

By Lem 5.7, it is enough to show that every L^1-bounded non-negative supermartingale converges a.s.

Let $\{X_n\}$ be a non-negative L^1-bounded supermartingale.

Suppose on the contrary that $\{X_n\}$ does not converge on a set with positive prob. Then $\exists$ $0<\alpha<\beta<\infty$ such that

$$E = \left\{ \omega : \ \underline{\lim_n} X_n(\omega) \quad \alpha < \beta \leq \overline{\lim_n} X_n(\omega) \right\}$$

has positive prob.

Define a sequence of stopping times $\{\tau_i\}_{i=0}^{\infty}$ by

$$\tau_0 \equiv 1,$$

$$\tau_{2i+1}(\omega) = \inf\{ n > \tau_{2i}(\omega) : \ X_n(\omega) > \beta \} \quad \text{if } i \geq 0 .$$

and

$$\tau_{2i}(\omega) = \inf\{ n > \tau_{2i-1}(\omega) : \ X_n(\omega) < \alpha \} \quad \text{if } i \geq 1,$$

where we $\inf \phi = +\infty$. Then

$$\tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots .$$

Let $P_j = P\{\omega : \tau_j(\omega) < \infty\}$. Then we have $P_{2j} \leq P_{2j-1}$.

Moreover

$$\begin{aligned}
\beta P_{2i+1} &\leq \int_{\{\tau_{2i+1} < \infty\}} X_{\tau_{2i+1}} \, dP = \int X_{\tau_{2i+1}} \, dP \\
&\leq \int X_{\tau_{2i}} \, dP \\
&= \int_{\{\tau_{2i} < \infty\}} X_{\tau_{2i}} \, dP \\
&\leq \alpha P_{2i}
\end{aligned}$$

Hence

$$P_{2i+1} \leq \frac{\alpha}{\beta} P_{2i} \leq \frac{\alpha}{\beta} P_{2i-1} \leq \cdots \leq \left(\frac{\alpha}{\beta}\right)^i P_1 \to 0 .$$

But $P_{2i+1} \geq P(E) > 0$, leading to a contradiction.

$\blacksquare$

Remark: • Under the assumption of Thm 5.6, we only have $\{X_n\}$ converges a.e.

• It is possible that X_n does not converge in L^1.

Def. A collection of r.v.'s X_i, $i \in \mathcal{I}$, is said to be uniformly integrable if

$$\lim_{M \to \infty} \left(\sup_{i \in \mathcal{I}} \int_{\{|X_i| > M\}} |X_i| \, dP \right) = 0 \qquad (2)$$

• If we pick M large enough so that the $\sup < 1$, then

$$E|X_i| = \int_{\{|X_i| > M\}} |X_i| \, dP + \int_{\{|X_i| < M} |X_i| \, dP$$

$$\leq 1 + M \quad \text{for all } i \in \mathcal{I}.$$

Thm 5.9 Let $\{X_n\}$ be a uniformly integral submartingale. Then it converges a.s and in L^1.

Pf. It follows from Thm 5.6 and the Vitali convergence Thm. ∎