Math 6261 23-03-24

Review.

Thm. Suppose that $X_1, X_2, \cdots$, are independent with mean 0.

If $\sum_{n=1}^{\infty} Var(X_n) < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges with prob. 1

- The following result, called the three-series Thm, provides necessary and sufficient conditions for the convergence of ΣX_n in terms of the individual distribution of X_n.

For $c>0$, let $X_n^{(c)} = X_n \mathbb{1}_{\{|X_n| \le c\}}$ be X_n truncated at c

Thm 4.6 (Kolmogrov's three-series Thm). Let $X_1, X_2, \cdots$, be independent r.v.'s. Let $c>0$. Then in order that $\sum_{n=1}^{\infty} X_n$ converges a.s., it is necessary and sufficient that

(1) $\sum_{n=1}^{\infty} P(|X_n| > c) < \infty$,

(2) $\sum_{n=1}^{\infty} E X_n^{(c)} < \infty$,

(3) $\sum_{n=1}^{\infty} Var(X_n^{(c)}) < \infty$.

Pf of Sufficiency: Suppose that ~~the~~ the three series converge

Write $m_n = E X_n^{(c)}$. By Thm 4.4,

$$\sum_{n=1}^{\infty} (X_n^{(c)} - m_n) \text{ converges a.s.}$$

Since $\sum_{n=1}^{\infty} m_n$ converges,

it follows that $\sum_{n=1}^{\infty} X_n^{(c)}$ converges a.s.

Since $\sum_{n=1}^{\infty} P(|X_n| > c) < \infty$,

by the first Borel-Cantelli lemma,

$$P(|X_n| > c \ \text{i.o}) = 0,$$

which implies that

$$P(X_n \neq X_n^{(c)} \ \text{i.o.}) = 0.$$

Hence it follows that $\sum_{n=1}^{\infty} X_n$ converges a.s.

Pf of necessity.

Suppose that $\sum_{n=1}^{\infty} X_n$ converges a.s. Fix $c > 0$.

Since $X_n \to 0$ a.s, it follows that $\sum X_n^c$ converges a.s.

and by the second Borel Cantelli lemma,

$$\sum_{n=1}^{\infty} P(|X_n| > c) < \infty.$$

In what follows we show that $\sum_{n=1}^{\infty} \mathrm{Var}(X_n^{(c)}) < \infty$

Write $M_n = E\left(\sum_{k=1}^{n} X_n^{(c)} \right)$

$$s_n^2 = \mathrm{Var}\left(\sum_{k=1}^{n} X_n^{(c)} \right).$$

Assume, on the contrary, that $S_n \to \infty$ as $n \to \infty$.

Since $X_n^{(c)} - E(X_n^{(c)})$ are uniformly bounded, by the CLT,

$$\lim_{n\to\infty} P\left(x < \frac{X_1^c + \cdots + X_n^c - M_n}{S_n} < y \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_x^y e^{-z^2/2} \, dz. \qquad (**)$$

Since $\sum_{n=1}^{\infty} X_n^c$ converges, $S_n \to \infty$ also implies that

$$\frac{X_1^c + \cdots + X_n^c}{S_n} \to 0 \quad \text{a.s.}$$

Hence

$$P\left(\left| \frac{X_1^c + \cdots + X_n^c}{S_n} \right| > \varepsilon \right) \to 0 \qquad (***)$$

Now $(**)$ and $(***)$ stand in contradiction: Since

$$P\left(x < \frac{X_1^c + \cdots + X_n^c - M_n}{S_n} < y, \quad \left| \frac{X_1^c + \cdots + X_n^c}{S_n} \right| < \varepsilon \right)$$

is positive for all sufficiently large n (if $x < y$), but then

$$x - \varepsilon < \frac{-M_n}{S_n} < y + \varepsilon. \qquad (****)$$

However, $(****)$ can not hold simutaneously for say, $(x-\varepsilon, y+\varepsilon) = (-1, 0)$ and $(x-\varepsilon, y+\varepsilon) = (1, 2)$.

Hence we have $\lim_{n\to\infty} S_n < \infty$, that is, $\sum_{n=1}^{\infty} Var(X_n^c) < \infty$

Now by Thm 4.4,

$$\sum_{n=1}^{\infty} (X_n^c - E X_n^c) \quad \text{converges. a.s.}$$

Since $\sum_{n=1}^{\infty} X_n^c$ converges a.s, we have $\sum_{n=1}^{\infty} E X_n^c$ converges. $\boxtimes$

Now we turn to the properties of random walks.

Def. (Random walks).

Let $X_1, X_2, \cdots$, be i.i.d. random vectors in $\mathbb{R}^d$. Set

$$S_n = X_1 + \cdots + X_n.$$

Then S_n is called a random walk.

For convenience, we build a new prob. space

$$\Omega = \{\omega = (\omega_1, \omega_2, \cdots) : \omega_i \in \mathbb{R}^d\}$$

$$\mathcal{F} = \mathcal{B}(\mathbb{R}^d) \times \cdots \times \mathcal{B}(\mathbb{R}^d) \times \cdots$$

$$P = \mu \times \mu \times \cdots \times \mu \cdots$$

where μ is the law of X_i.

Set $X_n(\omega) = \omega_n$.

Let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of positive integers.

Def. A finite permutation $\pi: \mathbb{N} \to \mathbb{N}$ is a map such that

$$\pi(i) \neq i$$

for only finitely many i.

For a finite permutation π of $\mathbb{N}$ and $\omega \in \Omega$, define

$$(\pi\omega)_i = \omega_{\pi(i)}.$$

That is, the coordinates of ω are rearranged according to π.

Def. An event $A \subset \Omega$ is called permutable if

$$P(A) = P(\pi^{-1}A)$$

for any finite permutation π.

In other words, A is permutable if the occurrence of A is not affected if re-arranging finitely many of the r.v.s

Def The collection of all permutable events is a σ-field.
It is called the exchangeable σ-field and is denoted by $\mathcal{E}$.

Example: Suppose $d=1$. Then

① $\{\omega: \quad S_n(\omega) \in B \quad \text{i.o.}\}$

② $\{\omega: \quad \limsup_{n\to\infty} \frac{S_n(\omega)}{c_n} \geq 1\}$

are permutable. This is because for each finite permutation π,

$$S_n(\omega) = S_n(\pi\omega) \quad \text{for large } n.$$

Fact: All events in the tail σ-field are permutable.

To see it, notice that if $A \in \sigma(X_{n+1}, X_{n+2}, \cdots)$, then the occurrence of A is unaffected by a permutation of $X_1, X_2, \cdots, X_n$.

The following result generalizes Kolmogrov's 0-1 Law:

Thm 4.7 (Hewitt-Savage 0-1 law).

If $X_1, X_2, \cdots,$ are i.i.d and $A \in \mathcal{E}$. Then

$$P(A) = 0 \text{ or } 1.$$

Pf. Let $A \in \mathcal{E}$.

The idea is to show that A is independent itself.

Since $A \in \sigma(X_1, X_2, \cdots)$, there exists $A_n \in \sigma(X_1, X_2, \cdots, X_n)$ such that

$$P(A_n \triangle A) \to 0,$$

Here $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.

Notice that A_n can be written as

$$A_n = \{\omega : (\omega_1, \omega_2, \cdots, \omega_n) \in B_n\}$$

for some B_n.

Now define a permutation π by

$$\pi(j) = \begin{cases} j+n & \text{if } \quad 1 \le j \le n \\ j-n & \text{if } \quad n+1 \le j \le 2n \\ j & \text{if } \quad j > 2n. \end{cases}$$

Then $P = P \circ \pi^{-1}$

$$P(A_n \triangle A) = P\{\omega : \omega \in A_n \triangle A\}$$

$$= P\{\omega : \pi\omega \in A_n \triangle A\}$$

$$= P\{\omega : \omega \in \pi^{-1}A_n \triangle \pi^{-1}A\}$$

$$= P\{\omega : \omega \in \pi^{-1}A_n \triangle A\}, \quad \text{since } \pi^{-1}A = A.$$

Write $A_n' = \pi^{-1}A_n$.

Then $A_n' = \{\omega : (\omega_{n+1}, \cdots, \omega_{2n}) \in B_n\}$.

So A_n and A_n' are independent.

Now $P(A_n \triangle A) \to 0$

$P(A_n' \triangle A) \to 0$

Since $A_n \triangle A_n' \subset (A_n \triangle A) \cup (A_n' \triangle A)$

So

$P(A_n \triangle A_n') \to 0$.

Notice that

$$0 \leq P(A_n) - P(A_n \cap A_n')$$

$$\leq P(A_n \cup A_n') - P(A_n \cap A_n')$$

$$= P(A_n \triangle A_n') \to 0$$

So $P(A_n \cap A_n') \to P(A)$.

But $P(A_n \cap A_n') = P(A_n)\, P(A_n') \to P(A)^2$.

It follows that

$$P(A) = P(A)^2. \qquad \boxed{}.$$

As an application, we have the following.

Thm 4.8: For a random walk on $\mathbb{R}$, there are only 4 possible cases, one of which has probability 1.

(i) $S_n = 0$ for all n.

(ii) $S_n \to +\infty$.

(iii) $S_n \to -\infty$

(iv) $-\infty = \liminf S_n < \limsup S_n = \infty$.

Pf. By Thm 4.7.

$$\limsup S_n = c \in [-\infty, \infty] \quad \text{a.s.}$$

Let $S_n' = S_{n+1} - X_1$. Since $S_{n+1} - X_1$ has the same distribution,

It follows that $c = c - X_1$.

If c is finite, then $X_1 \equiv 0$ and (i) occurs.

If c is not finite, then $c = +\infty$ or $-\infty$.

The same analysis applies to the liminf.

$\blacksquare$.