Beltrami Holomorphic Flow (Bojarski)

Theorem: (Beltrami Holomorphic flow on S^2) There is a one-to-one correspondence between the set of quasiconformal diffeomorphisms of S^2 that fix the points 0, 1 and ∞ and the set of smooth complex-valued functions μ on S^2 with $\|\mu\|_\infty = k < 1$.

(Here, we identify S^2 with extended complex plane $\overline{\mathbb{C}}$).

Also, the solution to the Beltrami's eqt depends holomorphically on μ. Let $\{\mu(t)\}$ be a family of Beltrami coefficient depending on a real / complex t. Let $\mu(t)(z) = \mu(z) + t\nu(z) + t\varepsilon(t)(z)$ for $z \in \mathbb{C}$ and $\|\varepsilon(t)\|_\infty \to 0$ as $t \to 0$. Then: for $\forall w \in \mathbb{C}$, $\nu, \varepsilon(t) \in L^\infty(\mathbb{C})$

$$f^{\mu(t)}(w) = f^{\mu}(w) + t V(f^\mu, \nu)(w) + o(|t|)$$ locally uniformly on $\mathbb{C}$

as $t \to 0$, where $$V(f^\mu, \nu)(w) = -\frac{f^\mu(w)(f^\mu(w)-1)}{\pi} \int_{\mathbb{C}} \frac{\nu(z)\left((f^\mu)_z(z)\right)^2 \, dx\,dy}{f^\mu(z)(f^\mu(z)-1)(f^\mu(z)-f^\mu(w))}$$

Proceed to construct:

$$
\begin{tikzcd}
\tilde{\mu}_0 \arrow[r] \arrow[d, leftrightarrow] & \tilde{\mu}_1 \arrow[r] \arrow[d, leftrightarrow] & \tilde{\mu}_2 \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \tilde{\mu}_k \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \tilde{\mu}_N = \tilde{\mu} \arrow[d, leftrightarrow] \\
\underset{= \mathrm{Id}}{\tilde{f}^{\tilde{\mu}_0}} \arrow[r] & \tilde{f}^{\tilde{\mu}_1} \arrow[r] & \tilde{f}^{\tilde{\mu}_2} \arrow[r] & \cdots \arrow[r] & \tilde{f}^{\tilde{\mu}_k} \arrow[r] & \cdots \arrow[r] & \tilde{f}^{\tilde{\mu}_N} = \tilde{g}
\end{tikzcd}
$$

Note that the above diagram can be realized iteratively by:

$$
\begin{cases}
\tilde{f}^{\tilde{\mu}_0} = \mathrm{Id} \\
\tilde{f}^{\tilde{\mu}_{k+1}} = \tilde{f}^{\tilde{\mu}_k} + \vec{V}\left(\tilde{f}^{\tilde{\mu}_k}, \frac{\tilde{\mu}}{N}\right),
\end{cases}
$$

where
$$
\vec{V}(f^{\mu}, \nu) = -\int_{\mathbb{C}} \frac{f^{\mu}(w)\left(f^{\mu}(w) - 1\right)}{\pi} \left(\frac{\nu(z)\left((f^{\mu})_z(z)\right)^2}{f^{\mu}(z)\left(f^{\mu}(z) - 1\right)\left(f^{\mu}(z) - f^{\mu}(w)\right)} \right) dx\, dy
$$

In discrete case, $\vec{V}(f^{\mu}, \nu)$ can be discretized as: $\sum_{v} k(v, w) \overset{\text{(area)}}{A_v}$

Example: Let $\phi_1 : S_1 \to D$ and $\phi_2 : S_2 \to D$ be global conformal parameterizations of S_1 and S_2 respectively.

Want to find $f : D \to D$ s.t. f minimizes:

$$E(f) = \int_D (F_1(w) - F_2 \circ f(w))^2 + |\mu(f)(w)|^2 \, dw$$

It can be formulated in term of μ:

$$\tilde{E}(\mu) = \int_D \left(F_1(w) - F_2 \circ f^{\mu}(w)\right)^2 + |\mu(w)|^2 \, dw.$$

To iteratively minimize $\tilde{E}$, consider:

$$\frac{d}{dt}\Big|_{t=0} \tilde{E}(\mu + t\nu) = \int_D \frac{d}{dt}\Big|_{t=0} (F_1(w) - F_2 \circ f^{\mu + t\nu}(w))^2 + |\mu(w) + t\nu(w)|^2 dw$$

$$= -\left(\int_D 2(F_1 - F_2(f^{\mu})) \nabla F_2(f^{\mu}) \frac{d}{dt}\Big|_{t=0} f^{\mu + t\nu} - 2\mu \cdot \nu\right)$$

Let $\begin{pmatrix} A \\ B \end{pmatrix} = 2\left(F_1 - F_2(f^{\mu})\right) \nabla F_2$; $\mu = \mu_1 + i\mu_2 \equiv \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$; $\nu = \nu_1 + i\nu_2 \equiv \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$.

Also, we can write: $\vec{V}(f^{\mu}, \nu) \left(= \frac{d}{dt}\Big|_{t=0} f^{\mu + t\nu} \right) = \int_D \begin{pmatrix} G_1 \nu_1 + G_2 \nu_2 \\ G_3 \nu_1 + G_4 \nu_2 \end{pmatrix} dz$

Then:
$$\frac{d}{dt}\Big|_{t=0} E(\mu + t\nu) = -\left[\int_D \left[\left(\int_D \begin{pmatrix} A \\ B \end{pmatrix} \cdot \begin{pmatrix} G_1 \nu_1 + G_2 \nu_2 \\ G_3 \nu_1 + G_4 \nu_4 \end{pmatrix} dz \right) - 2\,\mu \cdot \nu \right] dw \right]$$

$$= -\int_D \left(\int_D \begin{pmatrix} AG_1 + BG_3 \\ AG_2 + BG_4 \end{pmatrix} dw - 2 \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right) \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} dz$$

$\therefore$ The descent direction for μ_1 is given by:

$$\frac{d\mu_1}{dt} = \int_D (AG_1 + BG_3)\, dw - 2\mu_1 \quad \text{and}$$

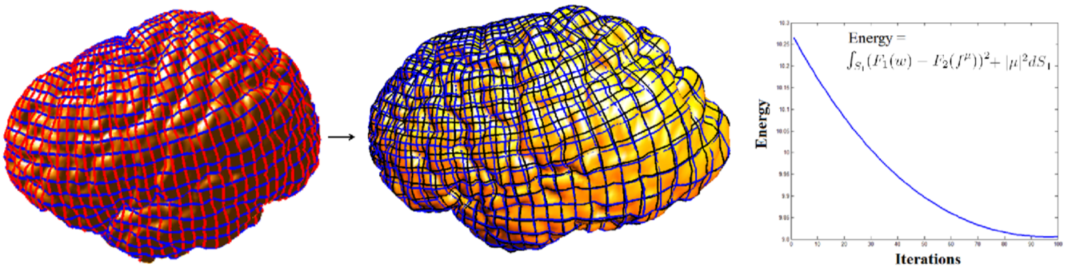
$$\frac{d\mu_2}{dt} = \int_D (AG_2 + BG_4)\, dw - 2\mu_2$$

$\therefore$ We can iteratively optimize the energy E as follows:

$$\mu^{n+1} = \mu^n + dt \underbrace{\begin{pmatrix} \int_D (A^n G_1^n + B^n G_3^n)\, dw - 2\mu_1^n \\ \int_D (A^n G_2^n + B^n G_4^n)\, dw - 2\mu_2^n \end{pmatrix}}_{\nu^n}$$

$$\begin{array}{ccc} \mu^n & \longrightarrow & \mu^{n+1} \\ \updownarrow & & \updownarrow \\ f^{\mu^n} & & f^{\mu^{n+1}} = f^{\mu^n} + \vec{V}(f^{\mu^n}, \nu^n) \end{array}$$

$\therefore$ We get (μ^n, f^{μ^n}) iteratively.



(A) Brain 1 (B) Brain 2 (C) Energy in each iteration

Fig. 7

ILLUSTRATION OF BHF OPTIMIZATION SCHEME ON BRAIN SURFACES. THIS EXAMPLE SHOWS THE OPTIMIZATION RESULT OF ATCHING TWO FEATURE FUNCTIONS F_1 AND F_2 ON THE TWO BRAIN SURFACES. THE BLUE GRID REPRESENTS THE INITIAL MAP, WHILE THE BLACK GRID REPRESENTS THE OPTIMIZED MAP.

with $F_1 \stackrel{def}{=} 5.2x^2 + 3.3y^2$

$F_2 \stackrel{def}{=} 6.8x^2 + 2.8y$

Theorem 4.2 (Beltrami holomorphic flow on $\mathbb{D}$) *There is a one-to-one correspondence between the set of quasiconformal diffeomorphisms of* $\mathbb{D}$ *that fix the points* 0 *and* 1 *and the set of smooth complex-valued functions* μ *on* $\mathbb{D}$ *for which* $\|\mu\|_\infty = k < 1$. *Furthermore, the solution* f^μ *depends holomorphically on* μ. *Let* $\{\mu(t)\}$ *be a family of Beltrami coefficients depending on a real or complex parameter* t. *Suppose also that* $\mu(t)$ *can be written in the form*

$$\mu(t)(z) = \mu(z) + t\nu(z) + t\epsilon(t)(z)$$

for $z \in \mathbb{D}$, *with suitable* μ *in the unit ball of* $C^\infty(\mathbb{D})$, $\nu, \epsilon(t) \in L^\infty(\mathbb{D})$ *such that* $\|\epsilon(t)\|_\infty \to 0$ *as* $t \to 0$. *Then for all* $w \in \mathbb{D}$

$$f^{\mu(t)}(w) = f^\mu(w) + tV(f^\mu, \nu)(w) + o(|t|)$$

locally uniformly on $\mathbb{D}$ *as* $t \to 0$, *where*

$$V(f^\mu, \nu)(w) = -\frac{f^\mu(w)(f^\mu(w)-1)}{\pi}\Bigg(\int_{\mathbb{D}} \frac{\nu(z)((f^\mu)_z(z))^2}{f^\mu(z)(f^\mu(z)-1)(f^\mu(z)-f^\mu(w))}\,dx\,dy$$

$$+\int_{\mathbb{D}} \frac{\overline{\nu(z)}\,\overline{((f^\mu)_z(z))}^2}{\overline{f^\mu(z)}(1-\overline{f^\mu(z)})(1-\overline{f^\mu(z)}f^\mu(w))}\,dx\,dy.\Bigg).$$

Lemma 7.1 *Let $f : \mathbb{D} \to \mathbb{D}$ be a diffeomorphism of the unit disk fixing* 0 *and* 1 *and satisfying the Beltrami equation $f_{\bar{z}} = \mu f_z$ with μ defined on $\mathbb{D}$. Let $\tilde{f}$ be the extension of f to $\overline{\mathbb{C}}$ defined as*

$$\tilde{f}(z) = \begin{cases} f(z), & \text{if } |z| \leq 1, \\ \frac{1}{\overline{f(1/\bar{z})}}, & \text{if } |z| > 1. \end{cases}$$

Then $\tilde{f}$ satisfies the Beltrami equation

$$\tilde{f}_{\bar{z}} = \tilde{\mu} \tilde{f}_z$$

on $\overline{\mathbb{C}}$, where the Beltrami coefficient $\tilde{\mu}$ is defined as

$$\tilde{\mu}(z) = \begin{cases} \mu(z), & \text{if } |z| \leq 1, \\ \frac{z^2}{\bar{z}^2} \overline{\mu(1/\bar{z})}, & \text{if } |z| > 1. \end{cases}$$

Speeding up Beltrami Holomorphic Flow

Key idea of BHF : Given f^{μ} with Beltrami coefficient μ,
$: \Omega_1 \to \Omega_2$

find f^{ν} with B.C. ν.

Let $\{\mu_n\}_{n=0}^{\infty}$ $\ni$ $\mu_0 = \mu$ and $\mu_\infty = \nu$. Proceed to modify f^{μ} iteratively:

$$\mu_0 = \mu \to \mu_1 \to \cdots \to \mu_n \to \cdots \to \mu_\infty = \nu$$

$$\updownarrow$$

$$f^{\mu_0} := f^{\mu} \to f^{\mu_1} \to \cdots \to f^{\mu_n} \to \cdots \to f^{\mu_\infty} = f^{\nu}$$

For a small variation from μ_0 to μ_1, we solve for f^{μ_1} $\ni$

$$f^{\mu_1} = \underset{f \in \underbrace{\mathrm{Diff}}_{\text{Space of diffeomorphism}}}{\operatorname{argmin}} \left\{ \left\| \frac{\partial f}{\partial \bar{z}} - \mu_1 \frac{\partial f}{\partial z} \right\|_2^2 \right\}.$$

Subject to $f^{\mu_1}\big|_{\partial\Omega_1} = \partial\Omega_2$

Let $f^{\mu_1} = f^{\mu_0} + g_1$. Reformulate as finding $g_1 : \Omega_1 \to \mathbb{R}^2 \ni$

$$(*) \quad g_1 = \operatorname*{argmin}_{g:\Omega_1 \to \mathbb{R}^2} \left\{ \left\| \frac{\partial (f^{\mu_0} + g)}{\partial \bar{z}} - \mu_1 \frac{\partial (f^{\mu_0} + g)}{\partial z} \right\|_2^2 \right\}$$

$$= \operatorname*{argmin}_{g:\Omega_1 \to \mathbb{R}^2} \left\{ \| A(\mu_1) g + A(\mu_1) f^{\mu_0} \|_2^2 \right\} \quad (*) \text{ subject to boundary constraint.}$$

where $A(\mu_1) \stackrel{\text{def}}{:=} \frac{\partial}{\partial \bar{z}} - \mu_1 \frac{\partial}{\partial z}$.

Discrete

Least Square Problem

Fast!!

(Instead of computing integration)

Algorithm 1 : *(Beltrami holomorphic flow)*
Input : $f^{\mu} : \Omega_1 \to \Omega_2$ *with BC* μ, *target BC* ν, *threshold* ϵ'
Output : *Sequence of quasi-conformal maps* $\{f^{\mu_n}\}_{n=1}^{\infty}$

1. *Set* $f^{\mu_0} = f^{\mu}$. *Solve Equation* ($*$) *to obtain* g_1;
2. *Given* f^{μ_n}, *compute* $\mu_n := \mu(f_n)$ *and* $\nu_n := (1-\epsilon)\mu_n + \epsilon\nu$; *solve Equation* ($*$) *to obtain* g_{n+1}; *Set* $f_{n+1} := f_n + g_{n+1}$;
3. *If* $\|\mu_{n+1} - \mu_n\| \geq \epsilon'$, *repeat step 2. Otherwise, stop the iteration.*

ADMM + BHF to solve Diffeomorphism Optimization Problem

Background: Minimize $\{ E_1(x) + E_2(Ax) \}$ where $A \in M_{m \times n}(\mathbb{R})$ has full column rank.

$x \in \mathbb{R}^n$

E_1, E_2 usually convex.

Reformulate: $(*)$ Minimize $\{ E_1(x) + E_2(y) \}$ subject to $Ax = y$.

Then: the augmented Lagrangian is given by:

$$L(x, y, \lambda, \mu) = E_1(x) + E_2(y) + \lambda^T (Ax - y) + \frac{\mu_k}{2} \|Ax - y\|^2$$

$(*)$ can be solved by:

$$\begin{cases} (x^{k+1}, y^{k+1}) = \operatorname{argmin} \{ L(x, y, \lambda^k, \mu^k) \} \\ \lambda^{k+1} = \lambda^k + \mu_k (A x^{k+1} - y^{k+1}) \end{cases}$$

Different choices of $\{\mu_k\}$ has been proposed to ensure convergence.

Penalty term

(*) can be difficult to solve.

Alternating direction method with multiplier (ADMM):

$$\begin{cases} x^{k+1} = \underset{x}{\operatorname{argmin}} \{ L(x, y^k, \lambda^k, \mu_k) \} \\ y^{k+1} = \underset{y}{\operatorname{argmin}} \{ L(x^{k+1}, y, \lambda^k, \mu_k) \} \\ \lambda^{k+1} = \lambda^k + \mu_k (Ax^{k+1} - y^{k+1}) \end{cases}$$

Remark: Once $\{\lambda^k\}$ and $\{\mu_k\}$ are carefully chosen, ADMM minimizes in few iterations.

Now, suppose we need to solve Diffeomorphism Optimization Problem:

$$f^* = \underset{f \in \mathrm{Diff}}{\operatorname{argmin}} \{ E_1(f) + E_2(\mu(f)) \} \text{ subject to:}$$

$$\|\mu(f^*)\|_\infty \overset{\text{def}}{=} \left\| \left(\frac{\partial f^*}{\partial \bar{z}}\right) \Big/ \left(\frac{\partial f^*}{\partial z}\right) \right\|_\infty < 1 .$$

e.g. Find $f^*: S_1 \to S_2$ such that it minimizes:

$$E(f) = \int_{S_1} |\mu(f)|^p + \alpha \int_{S_1} | H_1 - H_2(f) |^2$$

mean curvatures

Reformulate: Find $f^*: S_1 \to S_2$ and $\nu^*: S_1 \to \mathbb{C}$ $\ni$

$$(f^*, \nu^*) = \underset{\substack{f \in \mathrm{Diff} \\ \mu \in \mathcal{B}}}{\operatorname{argmin}} \{ E_1(f) + E_2(\mu) \} \ni \text{① } \nu^* = \mu(f^*)$$

$$\text{② } \|\nu^*\|_\infty < 1 .$$

Space of Beltrami coefficient

Augmented Lagrangian:

$$L(f,\nu,\lambda_{Re},\lambda_{Im},\rho) = E_1(f) + E_2(\nu) + \langle \lambda_{Re}, Re(\nu-\mu(f))\rangle + \langle \lambda_{Im}, Im(\nu-\mu(f))\rangle + \frac{\rho}{2}\|\nu-\mu(f)\|_2^2$$

where $\langle \alpha, \beta\rangle \overset{def}{=} \int_{S_1} \alpha\beta$ and $\|\alpha\| \overset{def}{=} \left(\int_{S_1} |\alpha|^2\right)^{1/2}$

Using ADMM:

$$f^{k+1} = \underset{f}{\mathrm{argmin}} \{ L(f, \nu_k, \lambda_{Re}^k, \lambda_{Im}^k, \rho^k) \} \quad — \quad 1$$

$$\nu^{k+1} = \underset{\nu}{\mathrm{argmin}} \{ L(f_{k+1}, \nu, \lambda_{Re}^k, \lambda_{Im}^k, \rho^k) \} \quad — \quad 2$$

λ_{Re}^k, λ_{Im}^k and ρ^k are updated as follows:

if $\|\nu^{k+1} - \mu(f^{k+1})\|_2 < \eta_k$, update:

$$\lambda_{Re}^{k+1} = \lambda_{Re}^k + \rho_k Re(\nu^{k+1} - \mu(f^{k+1}))$$

$$\lambda_{Im}^{k+1} = \lambda_{Im}^k + \rho_k Im(\nu^{k+1} - \mu(f^{k+1}))$$

$$\rho_{k+1} = \rho_k$$

if $\| \nu^{k+1} - \mu(f^{k+1}) \| \geq \eta_k$, update:

$$\lambda_{Re}^{k+1} = \lambda_{Re}^{k} \ ; \ \lambda_{Im}^{k+1} = \lambda_{Im}^{k}$$

$$\rho_{k+1} = \rho_k (1 + \gamma_k)$$

(η_k is chosen to be $\downarrow$
γ_k can be constant)

<u>Solving subproblem (2) involving ν:</u>

Very often, Euler-Lagrange eqt of $E_2(\mu)$ is an elliptic PDE. e.g. $\int_{S_1} |\nabla \mu|^2 + |\mu|^2$. Then, E-L eqt of (2) can be written as:

$$\underbrace{\Delta \mu - 2\mu - \lambda_{Re}^{k} \, Re(\nu - \mu(f^{k+1})) - \lambda_{Im}^{k} \, Im(\nu - \mu(f^{k+1})) - \rho_k (\nu - \mu(f^{k+1})) = 0}_{\text{Discretize} \rightsquigarrow \text{Sparse Symmetric positive definite linear system.}}$$

Solving subproblem ① involving f : $\tilde{\nu}$

$$f^{k+1} = \arg\min_{f} \left\{ E_1(f) + \langle \lambda_{Re}, Re(\nu - \mu(f)) \rangle + \langle \lambda_{Im}, Im(\nu - \mu(f)) \rangle + \frac{\rho}{2} \| \nu - \mu(f) \|_2^2 \right\}$$

$E_1(f)$ can be minimized using gradient descent algorithm.

e.g. $E_1(f) = \| I_1 - I_2(f) \|_2^2$

curvatures on S_1 and S_2

Then: descent direction $\vec{V}_1 = 2(I_1 - I_2(f)) \nabla f$

Last three terms can be minimized over B.C. to get a descent direction $\partial \tilde{\nu}$:

$$\partial \tilde{\nu} = -\lambda_{Re} - \lambda_{Im} + \rho(\nu - \mu(f))$$

After few iteration, we get a new B.C. $\tilde{\nu}$.

We can find the associated QC map $\tilde{f} = f^k + \vec{V}_2 \Rightarrow$

$$\frac{\partial \tilde{f}}{\partial \bar{z}} = \tilde{\nu} \frac{\partial \tilde{f}}{\partial z}$$

Overall, the descent direction to optimize :

$$\frac{df^k}{dt} = \vec{V}_1 + \vec{V}_2$$