Lecture 5:

Computation of discrete harmonic map

Let M be a triangulated surface. A piecewise linear function or map is a function/map on M such that it is linear on each triangular face.

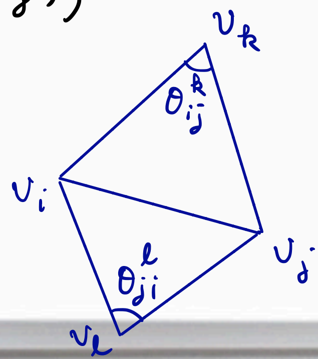
Theorem: Given a piecewise linear function $f: M \to \mathbb{R}$, then the harmonic energy of f is given by:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_i) - f(v_j)\right)^2$$

where

$$w_{ij} = \cot \theta_{ij}^{k} + \cot \theta_{ji}^{l}$$

(Cotangent formula)



Definition: (Laplace operator) The discrete Laplacian Δ_{PL} on a piecewise linear function f is

$$\Delta_{PL} f(v_i) = \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right)$$

Hence, if f minimizes the discrete harmonic energy, then:

$$\Delta_{PL} f \equiv 0$$

Remark: The motivation of this definition is by taking the derivative of the discrete harmonic energy:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right)^2$$

Recall: The Euler-Lagrange eqt of $\int_M |\nabla f|^2$ is given by $\Delta f = 0$.

Computational Algorithm for Disk Harmonic Maps

Input:A topological disk M;
Output:A harmonic map $\varphi : M \to \mathbb{D}^2$

1. Construct boundary map to the unit circle, $g : \partial M \to \mathbb{S}^1$, g should be a homeomorphism;
2. Compute the cotangent edge weight;
3. for each interior vertex $v_i \in M$, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_j \sim v_i} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0;$$

4. Solve the linear system, to obtain φ.

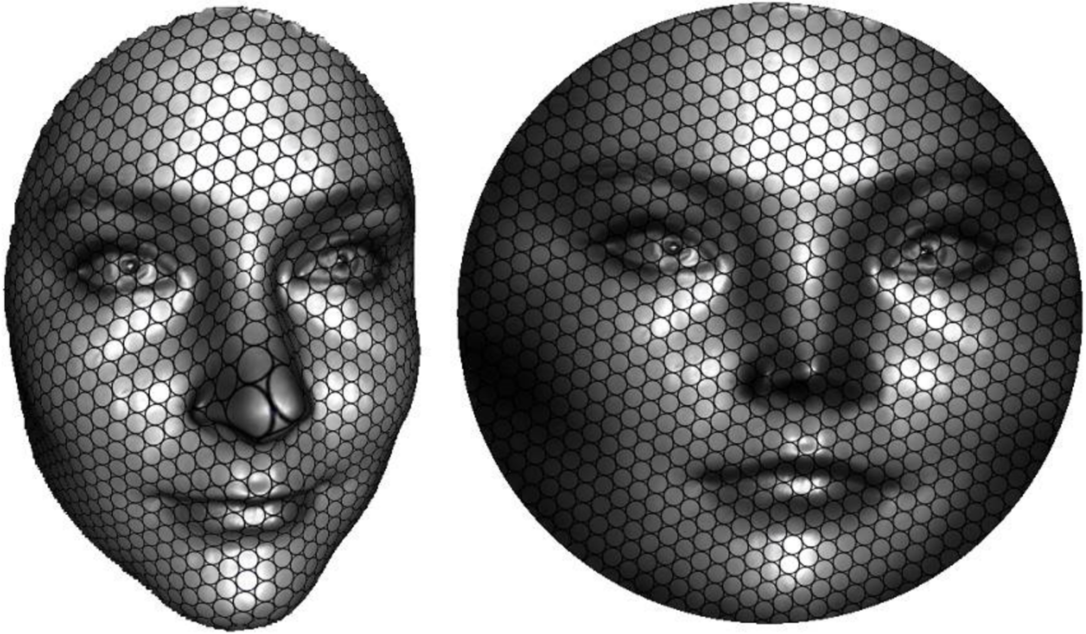


Figure: Harmonic map between topological disks.

Important fact:

Theorem: Suppose a harmonic map $\varphi: (S, g) \to \Omega \subseteq \mathbb{R}^2$ satisfies:

① Ω is convex;

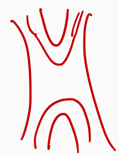
② the restriction of $\varphi: \partial S \to \partial \Omega$ on the boundary is homeomorphic

Then: u is diffeomorphic in the interior of S.

Proof: By regularity theory of harmonic maps, we get the smoothness of the harmonic map. Assume $\varphi: (x, y) \to (u, v)$ is not homeomorphic, then there is an interior point $p \in \Omega$, the Jacobian matrix of φ is degenerated at p.

$\therefore \exists\, a, b \in \mathbb{R}$ (not all zero) such that:

$$a \nabla u(p) + b \nabla v(p) = 0$$



By $\Delta u = \Delta v = 0$, the auxilary function

$$f(q) = a\,u(q) + b\,v(q) \text{ is also harmonic.}$$

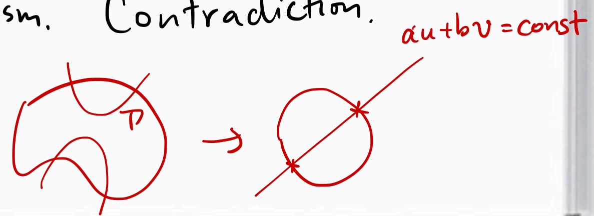
$\because \nabla f(p) = 0 \quad \therefore$ p is a saddle point of f.

Consider $\Gamma = \{ q \in S \mid f(q) = f(p) - \varepsilon \}$ (level set of f near p)

Γ has two connected components, intersecting ∂S at 4 points.

But Ω is a planar convex domain, $\partial\Omega$ and the line $au + bv = \text{const}$ have two intersection points. By assumption, $\varphi|_{\partial S}$ is a homeomorphism. Contradiction.

$\therefore$ φ is homeomorphic.



Theorem: If $f : S \to \Omega \subseteq \mathbb{R}^2$ and $g : S \to \Omega$ are
convex

both harmonic map satisfying $f|_{\partial S} = g|_{\partial S} = h$ (given),

then: $f \equiv g$.

Harmonic map v.s. genus-0 surface conformal map

Theorem: (Rado) Let $f: M \to N$ be a harmonic homeomorphism between genus-0 closed surfaces M and N. Then: f is a conformal map.

(Genus-0 : Harmonic $\Leftrightarrow$ conformal)

Remark: Allow us to compute spherical conformal parameterization by energy minimization (computable!)

Computation of genus-0 spherical conformal parameterization

Let $M = (V, E, F)$ be a triangulation mesh, which is of genus-0.

The harmonic energy of a discrete map $f : M \to \mathbb{S}^2$ (unit sphere) is given by:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right)^2$$

(Note that $\|f(v_i)\|^2 = 1$ for $\forall v_i \in V$)

We proceed to minimize $E(f)$ according to a nonlinear heat diffusion process:

$$\frac{d\vec{f}}{dt} = -\mathcal{D}\vec{f} \quad \left(\vec{f} = \begin{pmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{pmatrix} \in \mathbb{R}^{n \times 3} ; \ \mathcal{D}\vec{f} = \text{descent direction} \right)$$

<u>Definition:</u> The normal component of the Laplacian is:

$$\left(\Delta f(v_i)\right)^{\perp} = \langle \Delta f(v_i), \vec{n}(f(v_i)) \rangle \, \vec{n}(f(v_i))$$

↑ unit normal direction at $f(v)$

$$\Delta f(v) = \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i)\right) \in \mathbb{R}^3$$

The tangential component of the Laplacian is:

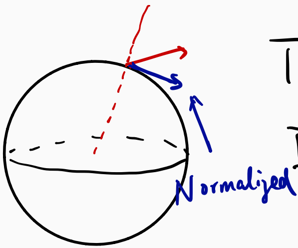
$$\left(\Delta f(v_i)\right)^{\parallel} = \Delta f(v_i) - \left(\Delta f(v_i)\right)^{\perp}.$$

According to gradient descent algorithm, the non-linear diffusion eqt is given by:

$$\frac{d\vec{f}(v,t)}{dt} = -\left(\Delta f(v)\right)^{\parallel}$$

Remark:

- $E(\vec{f})$ is minimized over $\vec{f}: V \to S^2$ or $\vec{f} \in M_{N\times 3}$, where N = # of vertices.



The descent direction $\vec{d}$ is in $\mathbb{R}^3$.

If we descend $E(\vec{f})$ along the descent direction without normalization, $\vec{f} + \Delta t \vec{d}$ may go outside S^2.

$\therefore$ Normalize $\vec{d}$ to the tangential direction on S^2

- $\vec{f} + \Delta t \vec{d}$ may not lie perfectly on S^2, we need to normalize again:

$$\frac{\vec{f} + \Delta t \vec{d}}{\|\vec{f} + \Delta t \vec{d}\|}$$

Computational Algorithm for Spherical Harmonic Map

Input: A genus zero closed mesh M;

Output: A spherical harmonic map $\varphi : M \to \mathbb{S}^2$;

1. Compute Gauss map $\varphi : M \to \mathbb{S}^2$, $\varphi(v) \leftarrow \mathbf{n}(v)$;
2. Compute the cotangent edge weight, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_i \sim v_j} w_{ij}(\varphi(v_j) - \varphi(v_i)),$$

3. project the Laplacian to the tangent plane,

normal of the sphere at $\varphi(v_i)$

$$D\varphi(v_i) = \Delta\varphi(v_i) - \langle \Delta\varphi(v_i), \varphi(v_i)\rangle\varphi(v_i)$$

4. for each vertex, $\varphi(v_i) \leftarrow \varphi(v_i) - \lambda D\varphi(v_i)$;

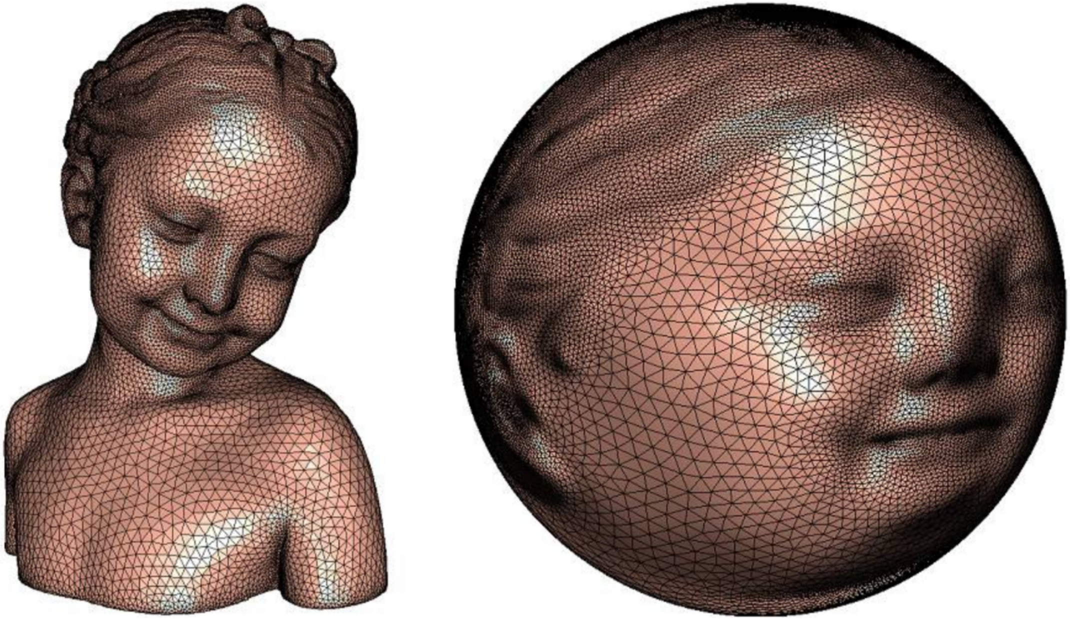
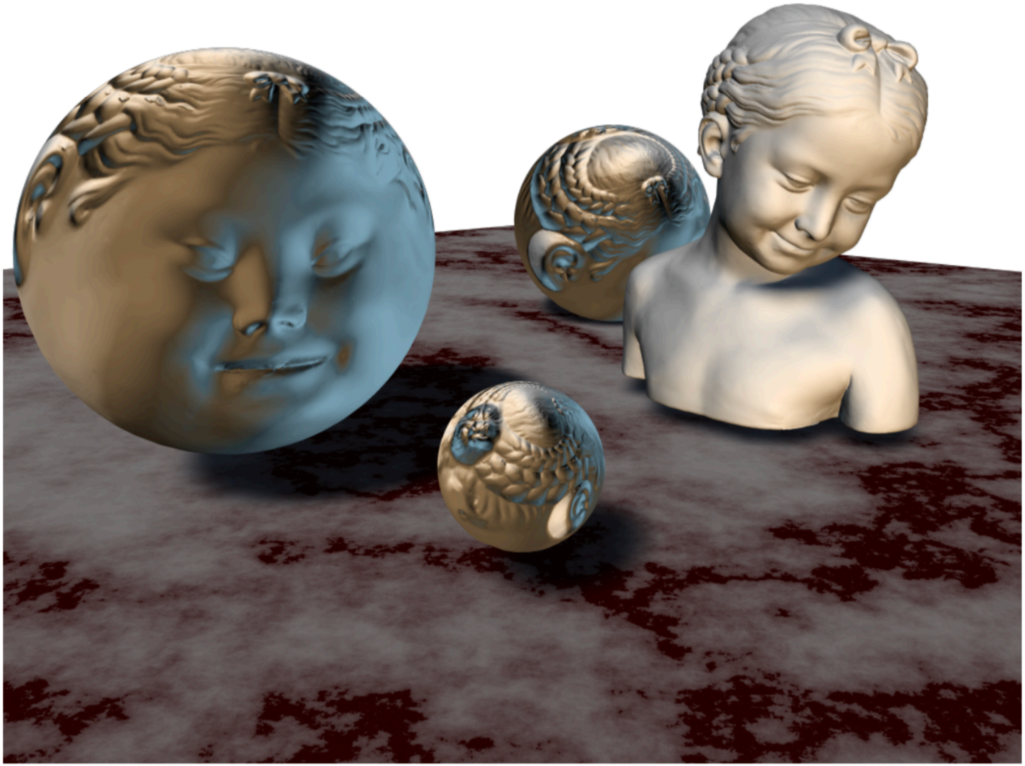


Figure: Harmonic map between topological spheres.



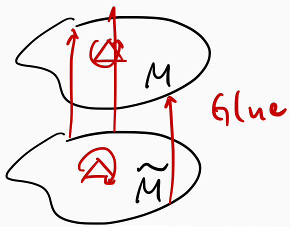
Computation of disk conformal parameterization

Goal: Given a simply-connected surface S, find $\phi : S \to \mathbb{D}$

Challenge: Cannot get it by computing harmonic map.
(Given a boundary homeomorphism $h : \partial S \to \partial \mathbb{D}$, there exist a unique harmonic map $H : S \to \mathbb{D} \ni H|_{\partial S} = h$. Amongst them, only few of them are conformal w/ suitable boundary conditions)

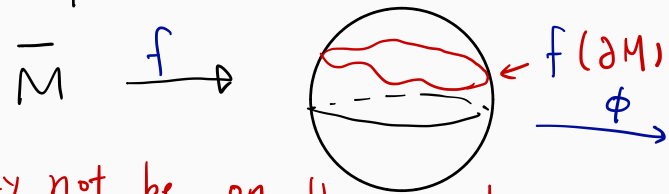
Idea: (Double covering)

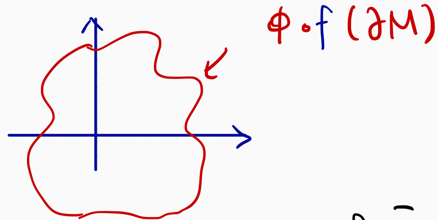
- Let $M = (V, E, F)$, construct $\tilde{M} = (V, \tilde{E}, \tilde{F})$ which is a "reflected" copy of M ($\tilde{F}$ has opposite orientation as F).
- Glue them along the boundary.
- $M \cup \tilde{M} = \bar{M}$ becomes a genus-0 closed surface



- $\tilde{M}$ can be parameterized onto $\mathbb{S}^2$ using the previous algorithm.

<u>Problem</u>:





$f(\partial M)$ may not be on the equator.

$\because$ $\bar{M}$ is a symmetric surface, $\exists$ a conformal map $h: \bar{M} \to \bar{\mathbb{C}}$ such that $h(\partial M) = \partial \mathbb{D}$.

<u>Solution</u>: Picking v_0, v_1, v_2 on ∂M. Reparameterize $\phi \circ f$ by a Mobius Transformation τ such that $h = \tau \circ \phi \circ f$ maps v_0, v_1, v_2 to $0, 1, i$ respectively.

e.g. $\underset{\mathbb{C}}{\underset{\cap}{z_0}}, \underset{\mathbb{C}}{\underset{\cap}{z_1}}, \underset{\mathbb{C}}{\underset{\cap}{z_2}}$ to $0, 1, \infty$ can be done by: $\tau_1 = \dfrac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)}$

etc ...

Input: A oriented surface with boundaries M;
Output: The double covering $\bar{M}$;

1. Make a copy of M, denoted as M';
2. Reverse the order of the vertices of each face of M';
3. Glue M and M' along their corresponding boundary edges to obtain $\bar{M}$.