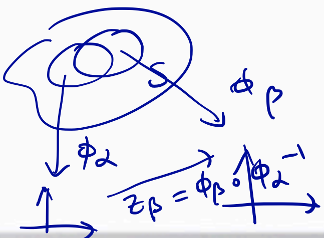
Computation of QC map using auxiliary metric

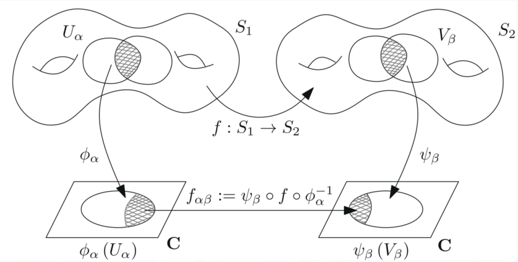
Definition: (Beltrami Differential) A Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ on a Riemann surface S is an assignment to each chart (U_α, ϕ_α) of an L^∞ complex-valued function μ_α defined on local parameters z_α such that :

$$\mu_\alpha(z_\alpha)\frac{dz_\beta}{dz_\alpha} = \mu_\beta(z_\beta)\frac{d\overline{z_\beta}}{d\overline{z_\alpha}}$$

on the domain which is also covered by another chart (U_β, z_β), where $\frac{dz_\beta}{dz_\alpha} = \frac{d}{dz_\alpha}\phi_{\alpha\beta}$ and $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$.



Definition: (QC map between Riemann surfaces) An orientation-preserving homeomorphism $f: S_1 \to S_2$ is called quasi-conformal associated with $\mu \frac{d\bar{z}}{dz}$ if for any chart (U_α, ϕ_α) on S_1 and for any chart (V_β, ψ_β) on S_2, the mapping $f_{\alpha\beta} := \psi_\beta \circ f \circ \phi_\alpha^{-1}$ is QC associated with $\mu_\alpha(z_\alpha)$. Also, on the domain on S_1 which is also covered by $(U_{\alpha'}, \phi_{\alpha'})$, $f_{\alpha'\beta} := \psi_\beta \circ f \circ \phi_{\alpha'}^{-1}$ is QC associated with $\mu_{\alpha'}(z_{\alpha'})$ where $\mu_{\alpha'}(z_{\alpha'}) = \mu_\alpha(z_\alpha)\left(\frac{d\bar{z}_\alpha}{d\bar{z}_{\alpha'}}\right) \Big/ \left(\frac{dz_\alpha}{dz_{\alpha'}}\right)$.



Theorem: (Auxilliary metric associated with a Beltrami Differential)

Suppose (S_1, g_1) and (S_2, g_2) are two metric surfaces, $f: S_1 \to S_2$ is a QC map associated with the Beltrami differential $\mu \frac{d\bar{z}}{dz}$.

Let z and w be the local isothermal coordinates of S_1 and S_2 respectively, indeed $g_1 = e^{2\lambda_1(z)} dz\, d\bar{z}$ and $g_2 = e^{2\lambda_2(w)} dw\, d\bar{w}$. Define an auxiliary Riemannian metric on S_1,

$$\tilde{g}_1 = e^{2\lambda_1(z)} |dz + \mu d\bar{z}|^2.$$

Then: the mapping $f: (S_1, \tilde{g}_1) \to (S_2, g_2)$ is a conformal mapping.

<u>Proof:</u> • Well-defined: Consider the region which is covered by two different charts z_α and z_β.

Suppose the local representations of g_1 under z_α and z_β are

$$e^{2\lambda_1(z_\alpha)} dz_\alpha \overline{dz_\alpha} \quad \text{and} \quad e^{2\lambda_2(z_\beta)} dz_\beta \overline{dz_\beta} \text{ respectively.}$$

$$\because \frac{dz_\alpha}{d\overline{z_\beta}} = 0 \quad \therefore \quad dz_\alpha = \frac{dz_\alpha}{dz_\beta} dz_\beta + \cancel{\frac{dz_\alpha}{d\overline{z_\beta}}}^{\,0}\, \overline{dz_\beta}$$

Thus,

$$\underbrace{e^{2\lambda_1(z_\alpha)} dz_\alpha \overline{dz_\alpha}}_{g_1} = e^{2\lambda_1(z_\alpha)} |dz_\alpha|^2 = \underbrace{e^{2\lambda_1(z_\alpha)} \left|\frac{dz_\alpha}{dz_\beta}\right|^2 |dz_\beta|^2}_{g_1}$$

$$= e^{2\lambda_2(z_\beta)} |dz_\beta|^2$$

$$\therefore \quad e^{2\lambda_2(z_\beta)} = e^{2\lambda_1(z_\alpha)} \left|\frac{dz_\alpha}{dz_\beta}\right|^2$$

$$\therefore\ e^{2\lambda_1(z_\alpha)} \left| dz_\alpha + \mu_\alpha d\bar{z}_\alpha \right|^2 = e^{2\lambda_1(z_\alpha)} \left| \frac{dz_\alpha}{dz_\beta} dz_\beta + \mu_\alpha \underbrace{\frac{d\bar{z}_\alpha}{d\bar{z}_\beta}}_{= \overline{\left(\frac{dz_\alpha}{dz_\beta}\right)}} d\bar{z}_\beta \right|^2$$

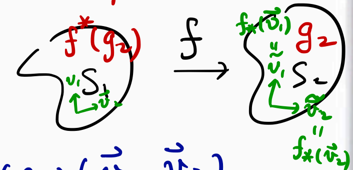
$$\left(\frac{d\bar{\phi}}{d\bar{z}} = \overline{\left(\frac{d\phi}{dz}\right)} \right)$$

$$= e^{2\lambda_1(z_\alpha)} \underbrace{\left| \frac{dz_\alpha}{dz_\beta} \right|^2}_{e^{2\lambda_2(z_\beta)}} \left| dz_\beta + \underbrace{\mu_\alpha \left(\frac{d\bar{z}_\alpha}{d\bar{z}_\beta}\right) \Big/ \left(\frac{dz_\alpha}{dz_\beta}\right)}_{= \mu_\beta} d\bar{z}_\beta \right|^2$$

$$= e^{2\lambda_2(z_\beta)} \left| dz_\beta + \mu_\beta d\bar{z}_\beta \right|^2$$

- Conformal under auxiliary metric:

Let $f^* g_2$ be the pull-back metric.



$$f^*(g_2)(\vec{v}_1, \vec{v}_2) \overset{\text{def}}{=} g_2(f_*(\vec{v}_1), f_*(\vec{v}_2))$$

Then: $$f^* g_2 = e^{2\lambda_2(f(z_\alpha))} \left| df(z_\alpha) \right|^2$$

(Check)

Now, under the pull-back metric, $f : (S_1, f^* g_2) \to (S_2, g_2)$ is isometric (length-preserving) inner-product preserving.

$$df(z_\alpha) = \frac{\partial f}{\partial z_\alpha} dz_\alpha + \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha$$

$$= \frac{\partial f}{\partial z_\alpha} \left(dz_\alpha + \mu_\alpha d\bar{z}_\alpha \right)$$

$$\therefore f^* g_2 = e^{2\lambda_2(f(z_\alpha))} \left| \frac{\partial f}{\partial z_\alpha} \right|^2 \left| dz_\alpha + \mu_\alpha d\bar{z}_\alpha \right|^2$$

$$\therefore f^* g_2 = e^{2(\lambda_2(f(z_\alpha)) - \lambda_1(z_\alpha))} \left| \frac{\partial f}{\partial z_\alpha} \right|^2 \tilde{g}_1$$

$$\tilde{g}_1 = e^{2\lambda_1(z_\alpha)} \left| dz_\alpha + \mu_\alpha d\bar{z}_\alpha \right|^2$$

$\therefore f^* g_2$ is conformal to $\tilde{g}_1$

$\therefore f : (S_1, \tilde{g}_1) \to (S_2, g_2)$ is conformal.

(V, E, F)

Discrete QC map

Definition: (Discrete metric) A discrete metric on a mesh M is a function $l: E \to \mathbb{R}^+$, such that on each triangle $[v_i, v_j, v_k]$, the triangle inequality holds:

$$l_{jk} + l_{ki} > l_{ij}$$

Definition: (Discrete conformal deformation) Let M be a triangulation mesh. Suppose l and L are different discrete metrics on M. L is a discrete conformal deformation of l if there exists a function $u: V \to \mathbb{R}$, called the discrete conformal factor, such that for all edges $[\vec{v}_i, \vec{v}_j] \in E$ on M,

$$L_{ij} = e^{u(v_i) + u(v_j)} l_{ij}$$

<u>Definition:</u> (Discrete local isothermal chart) Let M be a triangular mesh. A mesh M_α is called a submesh of M if every vertices, edges and faces of M_α belongs to M. A discrete local isothermal chart $(M_\alpha, \phi_\alpha : M_\alpha \to \mathbb{C})$ is a discrete conformal map from M_α to a mesh $\phi_\alpha(M_\alpha)$ embedded in $\mathbb{C}$.

<u>Definition:</u> (Discrete Beltrami Differential) A discrete Beltrami differential $\{\mu_\alpha\}_{\alpha \in I}$ is an assignment to each local isothermal chart (M_α, z_α) on M_α of a complex-valued function μ_α defined on every vertices of $\phi_\alpha(M_\alpha)$ with $\|\mu_\alpha\|_\infty < 1$ such that

$$\underbrace{\frac{\mu_\alpha(v_i)+\mu_\alpha(v_j)}{2}}_{=(\mu_\alpha)_{ij}} \underbrace{\frac{\overline{z_\alpha(v_j)-z_\alpha(v_i)}}{z_\alpha(v_j)-z_\alpha(v_i)}}_{=\frac{\overline{(dz_\alpha)_{ij}}}{(dz_\alpha)_{ij}}} = \underbrace{\frac{\mu_\beta(v_i)+\mu_\beta(v_j)}{2}}_{=(\mu_\beta)_{ij}} \underbrace{\frac{\overline{z_\beta(v_j)-z_\beta(v_i)}}{z_\beta(v_j)-z_\beta(v_i)}}_{=\frac{\overline{(dz_\beta)_{ij}}}{(dz_\beta)_{ij}}}$$

where $[v_i, v_j]$ is covered by both (M_α, z_α) and (M_β, z_β).

__Definition:__ (Discrete QC map) Let $\{\mu_\alpha\}$ be a given Beltrami differential. A mapping $f = (M_1, l) \to (M_2, L)$ between M_1 and M_2 (with the same connectivity) is a discrete __quasi-conformal map__, if with respect to a new metric $\tilde{l}$ on M_1, the mapping $f: (M_1, \tilde{l}) \to (M_2, L)$ is discrete conformal,

where
$$\tilde{l}_{ij} \overset{\text{def}}{=} l_{ij} \frac{|dz_{ij} + \mu_{ij} d\bar{z}_{ij}|}{|dz_{ij}|}$$

$$\left(dz_{ij} = z(v_j) - z(v_i), \quad \mu_{ij} = \frac{\mu(v_i) + \mu(v_j)}{2} \right)$$

$\tilde{l}$ is called the discret auxiliary metric

Well-defined?

Suppose an edge $[v_i, v_j]$ is covered by both charts z^α and z^β, we have:

$$l_{ij} \frac{|dz_{ij}^\alpha + \mu_{ij}^\alpha \overline{dz_{ij}^\alpha}|}{|dz_{ij}^\alpha|} = l_{ij} \left| 1 + \mu_{ij}^\alpha \frac{\overline{dz_{ij}^\alpha}}{dz_{ij}^\alpha} \right|$$

$$= l_{ij} \left| 1 + \mu_{ij}^\beta \frac{\overline{dz_{ij}^\beta}}{dz_{ij}^\beta} \right|$$

$$= l_{ij} \frac{|dz_{ij}^\beta + \mu_{ij}^\beta \overline{dz_{ij}^\beta}|}{|dz_{ij}^\beta|}$$

Well-defined!!

<u>Theorem:</u> Suppose (M_1, l) and (M_2, L) are two triangular meshes and $f: M_1 \to M_2$ is a discrete QC map with Beltrami differential $\{\mu_\alpha\}_{\alpha \in I}$. Under the auxilliary metric $\tilde{l}$ associated with $\{\mu_\alpha\}$, the mapping $f: (M_1, \tilde{l}) \to (M_2, L)$ is discrete conformal.

$\therefore$ Discrete QC $\rightsquigarrow$ Discrete conformal.

<u>Algorithm to compute QC map</u> Given μ

① Change the edge length l to $\tilde{l}$ such that

$$\tilde{l}_{ij} = \frac{|dz_{ij} + \mu_{ij} \overline{dz_{ij}}|}{|dz_{ij}|} l_{ij}$$

② Compute the angle structure of mesh based on $\tilde{l}_{ij}$

③ Use whatever existing algorithm to compute conf. map under the new angle structure.