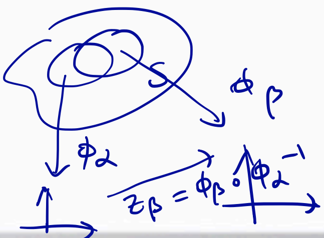
Computation of QC map using auxiliary metric

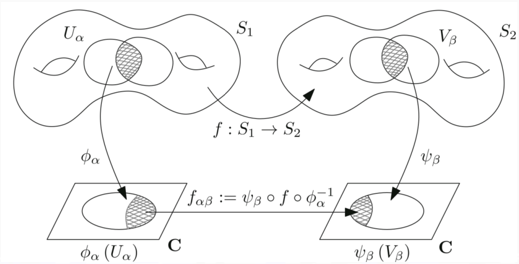
Definition: (Beltrami Differential) A Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ on a Riemann surface S is an assignment to each chart (U_α, ϕ_α) of an L^∞ complex-valued function μ_α defined on local parameters z_α such that:

$$\mu_\alpha(z_\alpha)\frac{d z_\beta}{d z_\alpha} = \mu_\beta(z_\beta)\frac{d\bar{z}_\beta}{d\bar{z}_\alpha}$$

on the domain which is also covered by another chart (U_β, z_β), where $\frac{dz_\beta}{dz_\alpha} = \frac{d}{dz_\alpha}\phi_{\alpha\beta}$ and $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$.



<u>Definition</u>: (QC map between Riemann surfaces) An orientation-preserving homeomorphism $f: S_1 \to S_2$ is called quasi-conformal associated with $\mu \frac{d\bar{z}}{dz}$ if for any chart (U_α, ϕ_α) on S_1 and for any chart (V_β, ψ_β) on S_2, the mapping $f_{\alpha\beta} := \psi_\beta \circ f \circ \phi_\alpha^{-1}$ is QC associated with $\mu_\alpha(z_\alpha)$. Also, on the domain on S_1 which is also covered by $(U_{\alpha'}, \phi_{\alpha'})$, $f_{\alpha'\beta} := \psi_\beta \circ f \circ \phi_{\alpha'}^{-1}$ is QC associated with $\mu_{\alpha'}(z_{\alpha'})$ where $\mu_{\alpha'}(z_{\alpha'}) = \mu_\alpha(z_\alpha)\left(\frac{d\bar{z}_\alpha}{d\bar{z}_{\alpha'}}\right) \Big/ \left(\frac{dz_\alpha}{dz_{\alpha'}}\right)$.



Theorem: (Auxilliary metric associated with a Beltrami Differential)
Suppose (S_1, g_1) and (S_2, g_2) are two metric surfaces, $f: S_1 \to S_2$ is a QC map associated with the Beltrami differential $\mu \frac{d\bar{z}}{dz}$.
Let z and w be the local isothermal coordinates of S_1 and S_2 respectively, indeed $g_1 = e^{2\lambda_1(z)} dz\, d\bar{z}$ and $g_2 = e^{2\lambda_2(w)} dw\, d\bar{w}$. Define an auxiliary Riemannian metric on S_1,

$$\tilde{g}_1 = e^{2\lambda_1(z)} |dz + \mu d\bar{z}|^2.$$

Then: the mapping $f: (S_1, \tilde{g}_1) \to (S_2, g_2)$ is a conformal mapping.