Topics in Algebra II: Coxeter groups

Felix Schremmer

April 26, 2023

By definition, a reflection group is a subgroup of the orthogonal group which is generated by a finite set of reflections. The concept of a Coxeter group is a purely group-theoretic analogue of those reflection groups. While the definition of a Coxeter group is fairly accessible, as it is a finitely presented group of a very specific shape, the theory is rather deep and still subject to research.

Typical examples of Coxeter groups are dihedral groups and symmetric groups. While the most important application of Coxeter groups is definitely Lie theory (coming from Weyl groups and affine Weyl groups), the theory is also relevant for classical geometry and knot theory.

The course will be divided into two parts. The first part will cover the fundamental structure theory of Coxeter groups, such as Bruhat order and parabolic subgroups. In the second part, we will turn our attention to more advanced topics related to Coxeter groups, depending on the participants' interests and backgrounds.

Please do not hesitate to email me any questions or suggestions for the courses contents, or these lecture notes in particular. Confer to the course website for the latest version of the lecture notes, as well as announcements of any kind.

Contents

1 Introduction **2**

2 Coxeter systems **4**

3 Examples of Coxeter groups **6**

4 The geometric representation **9**

5 Length **12**

6 Structure theory of Coxeter groups **16**

7 Matsumoto's Theorem **18**

8 Parabolic subgroups **21**

9 Bruhat order **23**

10 Bruhat order on parabolic quotients **26**

11 The Tableau criterion **28**

12 Finite Coxeter groups **30**

13 Weak Order **32**

14 Foldings **34**

15 List of finite Coxeter groups **37**

16 Iwahori-Hecke algebra **40**

17 Braid group **42**

18 Kazhdan-Lusztig polynomials **43**

19 Kazhdan-Lusztig representations **46**

20 Reflection orders **47**

21 Reflection subgroups **50**

22 Demazure products **53**

23 Finiteness results **54**

24 Conjugacy classes **56**

1 Introduction

The definition of a Coxeter group is fairly abstract, so let us motivate it by discussing the symmetric group.

Definition 1. Let $n \geqslant 0$ be an integer. The *symmetric group over n letters* is defined as the set of all bijective maps from the set $\{1, \ldots, n\}$ into itself. It becomes a group under function composition.

For $i \neq j$, we denote by $(i\ j) \in S_n$ the map that interchanges i and j and leaves all other elements of $\{1, \ldots, n\}$ fixed, called a *transposition.* The transpositions $s_i = (i\ i+1)$ are called *standard transpositions.* More generally, for pairwise distinct $i_1, \ldots, i_\ell$, we write

$$(i_1\ i_2\ \cdots\ i_\ell) \in S_n$$

for the ℓ-*cycle*, sending i_1 to i_2 etc.

Elements in $f \in S_n$ are written as a product of cycles, or in the two-row notation

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}.$$

Our main goal for now is to show the following:

Theorem 2. *The symmetric group S_n is generated by the standard transpositions $s_1, \dots, s_{n-1}$.*

This is certainly well-known and a combinatorial proof is not hard to come by. However, we want to give a *geometric* proof of Theorem 2.

In order to give such a geometric proof, we introduce a useful representation of the symmetric group.

Denote by V one of the following vector spaces:

$$V = \mathbb{R}^n \text{ or } V = \{(v_1, \dots, v_n) \in \mathbb{R}^n \mid v_1 + \cdots + v_n = 0\}.$$

We have an action of S_n on V by

$$f.(v_1, \dots, v_n) = (v_{f(1)}, \dots, v_{f(n)}), \qquad f \in S_n, (v_1, \dots, v_n) \in V.$$

For $i \neq j$, denote by $H_{i,j}$ the hyperplane

$$H_{i,j} = \{(v_1, \dots, v_n) \in V \mid v_i = v_j\} = \{v \in V \mid (i\ j).v = v\}.$$

Call a vector $v \in V$ *regular* if v does not lie on any of the hyperplanes $H_{i,j}$, i.e. if the coordinates are pairwise distinct. Denote by $V^{\text{reg}} \subset V$ the set of regular vectors.

The connected components of V^{reg} are called *Weyl chambers*. An example for such a Weyl chamber is the *dominant chamber*

$$C = \{(v_1, \dots, v_n) \in V \mid v_1 < \cdots < v_n\}.$$

It is easy to see that we get a bijective map

$$S_n \to \{\text{Weyl chambers}\}, \qquad f \mapsto fC.$$

(pictures)

We define the *length function* in S_n as follows: The length $\ell(f)$ of $f \in S_n$ is the number of hyperplanes $H_{i,j}$ such that C and fC lie on opposite sides of $H_{i,j}$. So $\ell(f) = 0$ if and only if $fC = C$, which means that f is the identity map. Note that S_n permutes the set of hyperplanes, where $fH_{i,j} = H_{f^{-1}(i), f^{-1}(j)}$.

Lemma 3. *Let $f \in S_n$ and $1 \leqslant i \leqslant n-1$ such $H_{i,i+1}$ lies between fC and C. Then $\ell(s_i f) = \ell(f) - 1$.*

Proof. We have

$$\begin{aligned}\ell(s_i f) =& \#\{H_{a,b} \mid H_{a,b} \text{ lies between } (s_i f)C \text{ and } C\} \\ =& \#\{H_{a,b} \mid H_{a,b} \text{ lies between } fC \text{ and } s_i C\} \\ =& \#\{H_{a,b} \mid H_{a,b} \text{ lies between } fC \text{ and } C \text{ and } H_{a,b} \neq H_{i,i+1}\} \\ =& \ell(f) - 1.\end{aligned}$$

□

Corollary 4. *An element $f \in S_n$ can be written as the product of $\ell(f)$ standard transpositions, but not as a product of less than $\ell(f)$ standard transpositions.*

Proof. Induction on $\ell(f)$. If $\ell(f) = 0$, then f must be the identity map, and there is nothing to prove.

If $\ell(f) > 0$, then $fC \neq C$. There must be *some* hyperplane between fC and C, and the only hyperplanes adjacent to C are those of the form $H_{i,i+1}$. So we find hyperplane $H_{i,i+1}$ between fC and C.

Now $\ell(s_i f) = \ell(f) - 1$ by the lemma, hence $s_i f$ can be written as the product of $\ell(f) - 1$ standard transpositions. We see that $f = s_i \cdot (s_i f)$ is a product of $\ell(f)$ standard transpositions.

For the converse, use the lemma to show that $\ell(s_i f) \leqslant \ell(f) + 1$ for all $f \in S_n$ and all simple reflections s_i. □

Observe that Theorem 2 is an immediate consequence of that corollary.

We have proved that S_n is generated by the s_i. Moreover, the following equations always hold true (and are easy to prove):

$$\begin{aligned}&s_i^2 = 1, \qquad s_i s_j = s_j s_i \text{ if } |i - j| \geqslant 2, \\ &s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.\end{aligned}$$

It turns out that these are enough: Any true equation consisting of s_i's on both sides can be derived from the above relations and the laws of group theory. By definition, this makes S_n an example of a Coxeter group.

Exercise 5. Consider the element

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3) \in S_3.$$

What is $\ell(f)$? Give an explicit formula writing f as the product of $\ell(f)$ transpositions.

2 Coxeter systems

Definition 6. A *Coxeter system* is a (typically finite) set S together with a map

$$m : S \times S \to \mathbb{Z}_{\geqslant 1} \cup \{+\infty\}$$

such that $m(s, s') = m(s', s)$ for all $s, s' \in S$, and $m(s, s') = 1$ if and only if $s = s'$.

The *Coxeter group* associated with a Coxeter system (S, m) is the group W with the following presentation:

Generators: The set S.

Relations: For each $s, s' \in S$, we require $(ss')^{m(s,s')} = 1$ unless $m(s, s') = +\infty$.

In the example of S_n, the set S is given by $\{s_1, \dots, s_{n-1}\}$ and the function m is

$$m(s_i, s_j) = \begin{cases} 1, & i = j, \\ 2, & |i - j| \geqslant 2, \\ 3, & |i - j| = 1. \end{cases}$$

We visualize the Coxeter system (S, m) by the *Coxeter diagram.* The nodes of the Coxeter diagram are in one to one correspondence with the set S. We draw an edge between s and s' iff $m(s, s') \geqslant 3$. If $m(s, s') \geqslant 4$, we moreover label the connecting edge by $m(s, s')$. The Coxeter diagram for S_n is known as A_{n-1}.

$$A_{n-1} : \bullet - \bullet - \cdots - \bullet.$$

Let us recall the concept of a finitely presented group in this specific context.

Write S^* for the set of *words* in S, i.e. finite sequences $(s_1, \dots, s_\ell)$ of arbitrary length ℓ with $s_i \in S$. An *elementary reduction* of a word w is the deletion of a subword of the form

$$\underbrace{(s, s', \dots, s, s')}_{\text{length } 2m(s,s')}$$

for $s, s' \in S$. Two words w, w' are *equivalent* if there exists a sequence of words $w = w_1, \dots, w_n = w'$ of words such that w_{i+1} is an elementary reduction of w_i or vice versa. Then W can be defined as the set of all equivalence classes of words. There is a group structure on W given by

$$[w]_\sim \cdot [w']_\sim = [w \circ w']_\sim,$$

the circle denoting composition of words. Moreover, we have a natural map $\varphi : S \to W$, sending an element $s \in S$ to the equivalence class of the one-letter word $[(s)]_\sim \in W$.

The group W is, up to unique isomorphism, uniquely determined by its *universal property*:

Proposition 7. *Let W' be any group, and $\psi : S \to W'$ any function such that for all $s, s' \in S$, we have*

$$(\psi(s)\psi(s'))^{m(s,s')} = 1 \text{ in } W'.$$

Then there exists a unique group homomorphism $\hat{\psi} : W \to W'$ such that $\psi = \hat{\psi} \circ \varphi$.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & W \arrow[d, dashed, "\hat{\psi}"] \\ & W'. \end{tikzcd}$$

It turns out that the presentation we gave for W is already minimal, i.e. that S is a minimal generating set and the set of relations is minimal as well.

Proposition 8. *The map $\varphi : S \to W$ is injective. For each $s, s' \in S$, the order of $\varphi(s)\varphi(s')$ in W is equal to $m(s, s')$.*

We will give a proof of this proposition later. For now, we use it as an excuse to write $s \in W$ instead of $\varphi(s) \in W$.

Exercise 9. If you are not familiar with the language of finitely presented groups, it makes sense to think a bit about these concepts:

(a) Going back to the construction of W as equivalence classes of words, why is the multiplication well-defined, and why does it make W a group?

(b) Give a proof of Proposition 7.

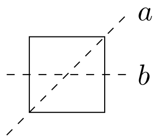
Exercise 10. Show that whenever $S \neq \varnothing$, there exists a subgroup $H \leqslant W$ of index 2 such that $\varphi(s) \notin H$ for all $s \in S$.

3 Examples of Coxeter groups

In view of Proposition 8, we say that (W, S) is a Coxeter group without specifying $m(\cdot, \cdot)$.

Example 11. The symmetric group S_n is a Coxeter group with respect to the standard transpositions $s_1, \ldots, s_{n-1}$.

Example 12. The Dihedral group of order $2n$ is defined to be the group of symmetries of the regular n-gon. e.g. D_6 is the symmetry group of an equilateral triangle. In D_{2n}, we find precisely n reflections. If a and b are "adjacent" reflections, then $(D_{2n}, \{a, b\})$ is a Coxeter group.



Example 13. The most interesting examples of Coxeter groups come from Lie theory. If G denotes a Lie group or a linear algebraic group, one may pick a *Borel subgroup* $B \subset G$ and denote by $W \subset G$ the[1] *Weyl group.* This is always a finite Coxeter group (for some choice of S determined by B). The *Bruhat decomposition*

$$G = \bigsqcup_{w \in W} BwB$$

is the first step towards understanding the group G (which is infinite and has non-trivial geometry) via the finite Coxeter group W.

For a concrete example, one may choose G to be the general linear group $G = \mathrm{GL}_n$, B the subgroup of upper triangular matrices and W the group of permutation matrices

[1] W is not actually a subset of G, but in practice, one can pretty much always treat it like a subgroup.

(i.e. matrices whose associated linear map permutes the set of standard basis vectors). Then $W \cong S_n$ is a Coxeter group. The Bruhat decomposition in this case is a fancy way to express Gauss' algorithm.

Example 14. Let G be the group of orthogonal $n \times n$-matrices with n odd. The Weyl group (W, S) is a Coxeter group with Coxeter diagram

$$B_{(n-1)/2} : \bullet \overset{4}{-} \bullet - \bullet - \cdots .$$

It can be identified with the subgroup of S_{n+1} consisting of those permutations w that satisfy

$$w(n+2-i) = n+2-w(i), \qquad i = 1, \ldots, n+1.$$

The simple reflections consist of $(i\ i+1)(n+1-i\ n+2-i)$ for $i = 1, \ldots, (n-1)/2$ and the one standard transposition interchanging $(n+1)/2$ with $(n+3)/2$.

Example 15. Let $n \geqslant 1$. We define W to be the group of matrices $g \in \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ subject to the following three conditions:

- Each row and each column of g contains only one non-zero entry.
- Each non-zero entry of g has the form t^m for some $m \in \mathbb{Z}$.
- The sum of those exponents occurring in g is equal to zero. Equivalently, $\det(g) = \pm 1$.

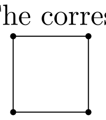
Since W contains all permutation matrices, we get a natural embedding of S_n into W. Group-theoretically, W is the semi-direct product of S_n acting on

$$\{(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \mid \mu_1 + \cdots + \mu_n = 0\}.$$

This group W is a Coxeter group with respect to the generating set $\{s_0, \ldots, s_{n-1}\}$. Here, $s_1, \ldots, s_n$ are the usual standard transpositions of $S_n \subset W$. The reflection s_0 is given by the matrix

$$\begin{pmatrix} 0 & & \ldots & 0 & t \\ & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & \\ t^{-1} & 0 & \ldots & & 0 \end{pmatrix}.$$

The corresponding Coxeter diagram is a cycle with n nodes:



The group W is known as the *affine Weyl group* of GL_n. Denote by L the field of formal Laurent series

$$L = k((t)) = \Big\{ \sum_{i=-N}^{\infty} a_i t^i \mid N \in \mathbb{Z} \text{ and } a_\bullet \in k \Big\}.$$

Let I denote the subgroup of $G = \{g \in \mathrm{GL}_n(L) \mid \det(g) \in (k[[t]])^\times\}$ consisting of those matrices $(g_{i,j}) \in L^{n\times n}$ such that

- Each $g_{i,j}$ lives in $k[[t]]$, i.e. has no negative powers of t occurring with non-zero coefficient.
- The elements above the main diagonal live in $tk[[t]]$, i.e. have no non-positive powers of t occurring with non-zero coefficient.

The group I is known as *Iwahori subgroup* of G. We get the *Iwahori-Bruhat decomposition*

$$G = \bigsqcup_{w\in W} IwI.$$

Similar decompositions exist for $\mathrm{SL}_n, \mathrm{GL}_n$ etc.

Exercise 16. Let (W_1, S_1) and (W_2, S_2) be two Coxeter groups.

(a) Show that the product group $W_1 \times W_2$ can be equipped with the structure of a Coxeter group ("is" a Coxeter group).

(b) Show that the free product group $W_1 * W_2$, i.e. the coproduct object in the category of groups, can be equipped with the structure of a Coxeter group.

Exercise 17. Give an example of two non-isomorphic Coxeter systems whose associated Coxeter groups are isomorphic.

Exercise 18. The Bruhat decomposition divides $\mathrm{GL}_3(k)$ into six subsets, as indexed by the six elements of S_3.

- For each $w \in S_3$, write down a list of conditions that determine whether a matrix $g \in \mathrm{GL}_3(k)$ lies in BwB or not.
- Determine the dimension of BwB/B as a variety over k (if k is algebraically closed) or manifold over k (if $k = \mathbb{R}$).

Exercise 19. Give a description of the Coxeter group $(D_\infty, \{a, b\})$ associated with the Coxeter graph

$$\bullet \overset{\infty}{-} \bullet.$$

4 The geometric representation

In this section, pick a Coxeter system (S, m), denote the corresponding Coxeter group by W and let $\varphi : S \to W$ be the canonical map.

It is easy to write down elements in W, namely by expressing them as $w = \varphi(s_1) \cdots \varphi(s_n)$ for some elements $s_1, \ldots, s_n \in S$. If two such expressions yield the same element $w \in W$, we can prove they are identical by applying the defining relations of a Coxeter group. But how can we prove that two expressions yield *distinct* elements in W?

We can answer this question by introducing a *group representation* of W, i.e. a suitable map $W \to \mathrm{GL}(V)$ for a vector space V. For each pair $s, s' \in S$, pick a real number $k_{s,s'} \in \mathbb{R}$ subject to the following conditions:

- $k_{s,s} = 2$ whereas $k_{s,s'} \leqslant 0$ for $s \neq s'$,
- $k_{s,s'} = 0$ if and only if $m(s, s') = 2$,
- $k_{s,s'} k_{s',s} = 4 \left(\cos \frac{\pi}{m(s,s')} \right)^2$ if $2 \leqslant m(s, s') < \infty$ and
- $k_{s,s'} k_{s',s} \geqslant 4$ if $m(s, s') = \infty$.

If is a canonical choice to always select

$$k_{s,s'} = -2 \cos \frac{\pi}{m(s, s')}.$$

In some circumstances, other choices might be more appealing (e.g. if we can chose all $k_{s,s'}$ to be integers).

We define V to be the $\mathbb{R}$-vector space with basis $\{\alpha_s \mid s \in S\}$. For each $s \in S$, we define the linear map $\alpha_s^\vee : V \to \mathbb{R}$ via

$$\alpha_s^\vee(\alpha_{s'}) = k_{s,s'}, \ s' \in S.$$

For $s \in S, v \in V$, define the reflection of v along α_s as

$$\sigma_s(v) := v - \alpha_s^\vee(v)\alpha_s.$$

Observe that σ_s defines a map in $\mathrm{GL}(V)$ such that $(\sigma_s)^2 = \mathrm{id}$.

If $v \in V$ is any vector, we write $v \geqslant 0$ if v is a $\mathbb{R}_{\geqslant 0}$-linear combination of the basis vectors $\alpha_s, s \in S$. Similarly, we write $v \leqslant 0$ if $-v \geqslant 0$.

Lemma 20. *Consider the rank 2 case $S = \{s, s'\}$ (the cardinality of the set S is known as the* rank *of the Coxeter system) and let $f : V \to V$ be the linear map defined by $f = \sigma_s \circ \sigma_{s'}$.*

(a) The order of f is equal to $m(s, s')$ (i.e. the n-fold composition f^n identity map if and only if $m(s, s')$ divides n).

(b) Let $g_n : V \to V$ be described as the alternating composition

$$g_n = \cdots \circ \sigma_{s'} \circ \sigma_s \circ \sigma_{s'}$$

of n terms ending with $\sigma_{s'}$. If $n < m(s,s')$, then $g_n(\alpha_s) \geqslant 0$.

Proof. Let $g = \sqrt{k_{s,s'}k_{s',s}}$ and let B denote the ordered basis $B = (\sqrt{g/k_{s,s'}}\alpha_s, \sqrt{g/k_{s',s}}\alpha_{s'})$ (resp. $B = (\alpha_s, \alpha_s')$ if $m(s,s') = 2$). With respect to the basis B, the matrix representing f is given by

$$\begin{pmatrix} -1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g & -1 \end{pmatrix} = \begin{pmatrix} g^2 - 1 & -g \\ g & -1 \end{pmatrix} =: A.$$

The determinant of A is one and the trace is given by $g^2 - 2$.

In case $m(s,s') = \infty$, we get $g \geqslant 2$. It follows that the matrix A has two positive real eigenvalues and is not conjugate to the identity matrix. In particular, f must have infinite order.

In case $m(s,s') < \infty$, the trace is further computed to be

$$4\left(\cos\frac{\pi}{m(s,s')}\right)^2 - 2 = \left(e^{i\pi/m(s,s')} + e^{-i\pi/m(s,s')}\right)^2 - 2 = e^{2\pi i/m(s,s')} + e^{-2\pi i/m(s,s')}.$$

It follows that $e^{\pm 2\pi i/m(s,s')}$ are the two eigenvalues of A. Moreover, direct inspection in case $m(s,s') = 2$ shows that A is always diagonalizable, even if the two eigenvalues are both -1. The claim on the order of f follows from this.

We now prove (b). Once can see this from Euclidean geometry, choosing a basis of the Euclidean plane consisting of two unit length vectors with angle $\frac{\pi}{m(s,s')}$ between them. Then A can be seen as the composition of the associated mirror reflections, which is a rotation by the angle $\frac{2\pi}{m(s,s')}$. The statements can now be seen to follow from Euclidean geometry. For these notes, we give a purely algebraic proof that leaves no special case unchecked.

Denote the two (previously computed) eigenvalues of A by λ_1 and λ_2, and define integers

$$a_n = 1 + \sum_{k=1}^{n} (\lambda_1^k + \lambda_2^k).$$

Observe that $a_n \geqslant 0$ whenever $n < m(s,s')/2$. We claim that A^n has the form

$$A^n = \begin{pmatrix} a_n & * \\ * & -a_{n-1} \end{pmatrix}, \qquad 1 \leqslant n < m(s,s').$$

The claim for $n = 1$ is easily verified. In an inductive step, let us calculate

$$f^n(\alpha_{s'}) = \sigma_s \tilde{f}^{(n-1)}(-\alpha_{s'}),$$

where $\tilde{f} = \sigma_{s'} \circ \sigma_s$. Since everything is symmetric in s and s', we may assume that the analogous statement on $\tilde{f}^{(n-1)}$ has been proved and conclude that

$$\tilde{f}^{(n-1)}(-\alpha_{s'}) = a_{n-1}(-\alpha_{s'}) + (*)(-\alpha_s).$$

Application of σ_s does not change the $\alpha_{s'}$-coordinate, so that

$$f^n(\alpha_{s'}) \equiv -a_{n-1}\alpha_{s'} \pmod{\mathbb{R}\alpha_s},$$

proving the claim on the lower-right coefficient of A^n. Observe that $\operatorname{tr}(A^n) = \lambda_1^n + \lambda_2^n = a_n - a_{n-1}$, so that the claim on the top-left coefficient follows.

In the situation of (b), observe that

$$g_n = \begin{cases} \sigma_{s'} f^{(n-1)/2}, & n \text{ odd} \\ f^{n/2}, & n \text{ even.} \end{cases}$$

In any case, it follows that

$$g_n(\alpha_s) \in a_{\lfloor n/2 \rfloor}\alpha_s + \mathbb{R}\alpha_{s'}$$

with $a_{\lfloor n/2 \rfloor} \geqslant 0$ whenever $n < m(s,s')$. Suppose that $g_n(\alpha_s) \not\geqslant 0$, i.e. that we had

$$g_n(\alpha_s) = a_{\lfloor n/2 \rfloor}\alpha_s + c\alpha_{s'}, \qquad c < 0.$$

Then

$$\sigma_s g_n(\alpha_s) = (-a_{\lfloor n/2 \rfloor} - ck_{s,s'})\alpha_s + c\alpha_{s'}$$

has a negative coefficient for α_s. Since $\sigma_s g_n = g_{n\pm 1}$, we immediately get a contradiction unless $n = m(s,s') - 1$ and $\sigma_s g_n = g_{n+1}$. The latter condition means that n is odd, by definition of g_n. If $m(s,s')$ is even and $n = m(s,s') - 1$, observe that $A^{m(s,s')}$ must be diagonalizable with two eigenvalues equal to -1, thus $f^{m(s,s')} = -1$. We conclude $\sigma_s g_n(\alpha_s) = -\alpha_s$, showing that the $g_n(\alpha_s) = \alpha_s$. ☐

We return to the situation of a general Coxeter group.

Theorem 21. *The map $S \to \mathrm{GL}(V), s \mapsto \sigma_s$ extends uniquely to a group homomorphism $\sigma : W \to \mathrm{GL}(V)$.*

Proof. In view of (Proposition 7), we have to show that $f := (\sigma_s\sigma_{s'})^{m(s,s')} \in \mathrm{GL}(V)$ is the identity map for all $s, s' \in S$ with $m(s,s') < +\infty$. We saw in the previous lemma that $f(\alpha_s) = \alpha_s$ and $f(\alpha_{s'}) = \alpha_{s'}$.

Observe that the matrix

$$A := \begin{pmatrix} \alpha_s^\vee(\alpha_s) & \alpha_s^\vee(\alpha_{s'}) \\ \alpha_{s'}^\vee(\alpha_s) & \alpha_{s'}^\vee(\alpha_{s'}) \end{pmatrix}$$

has determinant $4 - k_{s,s'}k_{s',s} > 0$ since $m(s,s') < +\infty$. It follows that for each vector $v \in V$, we find real numbers $c, d \in \mathbb{R}$ such that $v' := v + c\alpha_s + d\alpha_{s'}$ satisfies

$$\alpha_s^\vee(v') = \alpha_{s'}^\vee(v') = 0.$$

Thus $f(v') = v'$ by definition of f. We conclude that $f(v) = v$. ☐

We will later see that this representation is faithful (i.e. an injective function). This allows us to replace (abstract) Coxeter group elements by concrete matrices.

Exercise 22. Show that σ is injective in the rank 2 case.

Exercise 23. In the situation of Lemma 20 (b), show that $g_n(\alpha_{s'}) \leqslant 0$ for $n < m(s, s')$. Show that $g_n(\alpha_s) \leqslant 0$ for $m(s, s') \leqslant n < 2m(s, s')$.

Exercise 24. Let $\sigma^\vee : W \to \mathrm{GL}(V^\vee)$ be the action of W on the dual space of V, given by the composing σ with the natural map $\mathrm{GL}(V) \to \mathrm{GL}(V^\vee)$. Show that for $s \in S$, the map $\sigma^\vee(s) : V^\vee \to V^\vee$ is given by

$$\sigma^\vee(s)(\lambda) = \lambda - \lambda(\alpha_s)\alpha_s^\vee.$$

5 Length

We saw that the map $\varphi : S \to W$ is injective, so let us write s instead of $\varphi(s)$ when denoting group elements. The elements of $s \in W$ are called *simple reflections*. We saw that each $w \in W$ can be written as

$$w = s_1 \cdots s_n$$

for some elements $s_1, \dots, s_n \in S$. In this case, we say that $(s_1, \dots, s_n)$ is a *word representing* w. A word that has minimal length among all words representing w is called a *reduced word* of w (also *reduced expression* and *reduced decomposition*). We define the *length* of w to be the length of any reduced word.

We have the important *triangle inequality*

$$\ell(w_1 w_2) \leqslant \ell(w_1) + \ell(w_2).$$

Moreover, the set S can be identified with the set of elements having length 1. It follows that $|\ell(ws) - \ell(w)| \leqslant 1$ (and in view of Exercise 10, it must be equal to 1).

Proposition 25. *Let $w \in W$ and $s \in S$. In case $\ell(ws) \geqslant \ell(w)$, we have $\sigma(w)(\alpha_s) \geqslant 0$. If $\ell(ws) \leqslant \ell(w)$, we have $\sigma(w)(\alpha_s) \leqslant 0$.*

Proof. We only show the first statement, as the second one follows from reversing the roles of w and ws (using $wss = w$).

If $\ell(w) = 0$, we must have $w = 1$ and there is nothing to show. So let $\ell(w) \geqslant 1$ and the statement be proved for all elements of smaller length.

If there exists a reduced word of w ending with s, then $\ell(ws) = \ell(w) - 1$, contradicting our assumption. Hence we find a reduced word of w ending in some $s' \neq s$. Put $w_1 := ws'$, so that

$$\sigma(w)(\alpha_s) = \sigma(w_1)\sigma(s')(\alpha_s) = \sigma(w_1)(\alpha_s) \underbrace{-k_{s',s}\sigma(w_1)(\alpha_{s'})}_{\geqslant 0 \text{ by induction}}.$$

If $\sigma(w_1)(\alpha_s) \geqslant 0$, we are done. Otherwise we get $\ell(w_1 s) < \ell(w)$ by induction. Define $w_2 := w_1 s$. If $\ell(w_2 s') < \ell(w_2)$, define $w_3 := w_2 s'$ and so forth. After at most $\ell(w)$ steps, we find an element w_n such that $w_n(\alpha_s), w_n(\alpha_{s'}) \geqslant 0$ and

$$w = w_n \underbrace{(\cdots s' s s')}_{n \text{ terms}}, \qquad \ell(w) = \ell(w_n) + n.$$

If $n < m(s,s')$, then $\sigma(\cdots s' s s')(\alpha_s) = c\alpha_s + d\alpha_{s'}$ for real numbers $c, d \geqslant 0$ by Lemma 20. Hence $\sigma(w)(\alpha_s) \geqslant 0$ as claimed.

If $n \geqslant m(s,s')$, we could write

$$\begin{aligned} ws &= w_n \underbrace{(\cdots s' s s' s)}_{n+1 \text{ terms}} \\ &\underset{\text{Def. Coxeter group}}{=} w_n \underbrace{(s' s s' \cdots)}_{2m(s,s')-n-1 \text{ terms}} . \end{aligned}$$

This implies

$$\ell(ws) \leqslant \ell(w_n) + 2m(s,s') - n - 1 < \ell(w_n) + n = \ell(w),$$

contradiction. $\square$

The *root system* of (W, S) is defined as

$$\Phi = \{\sigma(w)(\alpha_s) \mid w \in W, s \in S\} \subseteq V.$$

It follows from Proposition 25 that each root is either positive (i.e. $\geqslant 0$) or negative (i.e. $\leqslant 0$). Write Φ^+ for the set of positive roots and Φ^- for the set of negative roots.

For each $w \in W$, we define the *inversion set* of w to be

$$\operatorname{inv}(w) = \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}.$$

Theorem 26. *The cardinality of the set* $\operatorname{inv}(w)$ *is equal to* $\ell(w)$*. More explicitly, given any reduced word* $w = s_1 \cdots s_\ell$*, the roots*

$$\alpha_1 = \sigma(s_\ell \cdots s_2)(\alpha_{s_1}),\ \alpha_2 = \sigma(s_\ell \cdots s_3)(\alpha_{s_2}), \ldots, \alpha_\ell = \alpha_{s_\ell}$$

are pairwise distinct and $\operatorname{inv}(w) = \{\alpha_1, \ldots, \alpha_\ell\}$*.*

Proof. If $w = s_1 \cdots s_\ell$ is a reduced word, then for each $i \in \{1, \ldots, \ell\}$, the root $\alpha_i = \sigma(s_\ell \cdots s_{i+1})(\alpha_{s_i})$ must be positive with

$$\sigma(w)(\alpha_i) = \sigma(s_1 \cdots s_{i-1})(-\alpha_{s_i}) \leqslant 0$$

by Proposition 25. If it happens that $\alpha_i = \alpha_j$ for some $i < j$, then

$$\alpha_{s_i} = \sigma(s_{i+1} \cdots s_\ell)(\alpha_i) = \sigma(s_{i+1} \cdots s_\ell)(\alpha_j) \sigma(s_{i+1} \cdots s_{j-1})(-\alpha_{s_j})$$

contradicts the fact that $s_{i+1}\cdots s_j$ is a reduced word. Hence the α_i are pairwise distinct and w has at least $\ell(w)$ inversions.

Conversely, if $\alpha \in \Phi^+$ satisfies $w\alpha \in \Phi^-$, there must be an index $i \in \{1,\ldots,\ell\}$ such that

$$\sigma(s_i\cdots s_\ell)(\alpha) \in \Phi^-, \quad \sigma(s_{i+1}\cdots s_\ell)(\alpha) \in \Phi^+.$$

It suffices to show that the only positive root $\beta \in \Phi^+$ with $\sigma(s_i)(\beta) \in \Phi^-$ is $\beta = \alpha_{s_i}$. By definition of σ, the conditions $\beta \in \Phi^+$ and $\sigma(s_i)(\beta) \in \Phi^-$ imply that $\beta \in \mathbb{R}_{>0}\alpha_{s_i}$.

We claim that $s \in S$ and $w \in W$ are such that $\sigma(w)(\alpha_s)$ is of the form $c\alpha_{s_i}$ for some $c \in \mathbb{R}_{\geqslant 0}$, then $c = 1$, via induction on $\ell(w)$.

Choose some s' with $\ell(ws') < \ell(w)$ and write w as in Proposition 25 as

$$w = w_n \underbrace{(\cdots s'ss')}_{n \text{ terms}}, \qquad \sigma(w_n)(\alpha_s) \geqslant 0,\ \sigma(w_n)(\alpha_{s'}) \geqslant 0.$$

We may write $\sigma(\cdots s'ss')(\alpha_s) = c_s\alpha_s + c_{s'}\alpha_{s'}$ with $c_s, c_{s'} \geqslant 0$. Then $c_s\sigma(w_n)(\alpha_s)$ and $c_{s'}\sigma(w_n)(\alpha_{s'})$ are both $\geqslant 0$ and their sum has only a non-zero α_{s_i}-coefficient. It follows that

$$c_s\sigma(w_n)(\alpha_s), c_{s'}\sigma(w_n)(\alpha_{s'}) \in \mathbb{R}_{\geqslant 0}\alpha_{s_i}.$$

In particular, these two vectors are linearly dependent. Since α_s and $\alpha_{s'}$ are not, we get $c_s = 0$ or $c_{s'} = 0$. It follows that $\sigma(\cdots s'ss')(\alpha_s)$ is already a positive multiple of either α_s or $\alpha_{s'}$. In view of Lemma 20, it must be equal to α_s or $\alpha_{s'}$. We see that $\alpha_{s_i} = c\sigma(w)(\alpha_s)$ is equivalent to

$$\alpha_{s_i} \in \{c\sigma(w_n)(\alpha_s), c\sigma(w_n)(\alpha_{s'})\}.$$

In both cases, we conclude $c = 1$ by induction. This finishes the proof. ☐

Corollary 27. *The group homomorphism $\sigma : W \to \mathrm{GL}(V)$ is injective.*

Proof. If w lies in the kernel, $\mathrm{inv}(w) = \varnothing$ by definition. In view of the previous theorem, this means $\ell(w) = 0$, i.e. $w = 1$. ☐

We will in the future frequently omit the explicit map σ and simply write $w\alpha$ instead of $\sigma(w)(\alpha)$.

Lemma 28. *Let $w_1, w_2 \in W$. Then*

$$\ell(w_1w_2) = \ell(w_1) + \ell(w_2) - 2\#(\mathrm{inv}(w_1) \cap \mathrm{inv}(w_2^{-1})).$$

Proof. We calculate

$$\begin{aligned}
\#\operatorname{inv}(w_1w_2) =&\#\{\alpha \in \Phi^+ \mid w_1w_2\alpha \leqslant 0 \text{ and } w_2\alpha \leqslant 0\} \\
&+ \#\{\alpha \in \Phi^+ \mid w_1w_2\alpha \leqslant 0 \text{ and } w_2\alpha \geqslant 0\} \\
=&\#\{\alpha \in \Phi^+ \mid w_2\alpha \leqslant 0\} \\
&- \#\{\alpha \in \Phi^+ \mid w_1w_2\alpha \geqslant 0 \text{ and } w_2\alpha \leqslant 0\} \\
&+ \#\{\alpha \in \Phi \mid w_1w_2\alpha \leqslant 0 \text{ and } w_2\alpha \geqslant 0\} \\
&- \#\{\alpha \in \Phi^- \mid w_1w_2\alpha \leqslant 0 \text{ and } w_2\alpha \geqslant 0\} \\
=&\#\operatorname{inv}(w_2) - \#\{\beta \in \Phi^- \mid w_1\beta \geqslant 0 \text{ and } w_2^{-1}\beta \geqslant 0\} \\
&+ \#\operatorname{inv}(w_1) - \{\beta \in \Phi^+ \mid w_1\beta \leqslant 0 \text{ and } w_2^{-1}\beta \leqslant 0\} \\
=&\ell(w_1) + \ell(w_2) - 2\#(\operatorname{inv}(w_1) \cap \operatorname{inv}(w_2^{-1})).
\end{aligned}$$

□

We finish this section by giving a different perspective on the set of positive roots.

Definition 29. A *reflection* in W is any element conjugate to an element of S. The set of reflections is denoted by $T \subset W$.

Proposition 30. *There is a bijective map*

$$s_\bullet : \Phi^+ \to T$$

that can be evaluated as follows: If $\alpha \in \Phi^+$ is written as $\alpha = v(\alpha_s)$ with $v \in W$ and $s \in S$, then $s_\alpha = vsv^{-1}$.

Proof. Certainly every positive root is of the form $v\alpha_s$ for some $v \in W$ and $s \in S$. By definition, every $t \in T$ is of the form vsv^{-1} for some $v \in V$ and $s \in S$. Replacing v by vs does not change t but changes $v\alpha_s$ to its negative, so we always can find a v such that $t = vsv^{-1}$ and $v\alpha_s \geqslant 0$.

However, none of these representations are unique. We have to show that for all $v, v' \in W$ and $s, s' \in S$ with $v\alpha_s, v'\alpha_{s'} \geqslant 0$ that

$$v\alpha_s = v'\alpha_{s'} \iff vsv^{-1} = v's'(v')^{-1}.$$

Replacing (v, v') by $((v')^{-1}v, 1)$, we may assume that $v' = 1$. Now we evaluate for $x \in V$

$$vsv^{-1}x = \sigma(v) \circ \sigma_s \circ \sigma(v^{-1})(x) = x - \alpha_s^\vee(v^{-1}x)v\alpha_s.$$

Observe that $v\alpha_s \in \operatorname{inv}(vsv^{-1})$.

If $vsv^{-1} = s'$ is a simple reflection, it follows from Theorem 26 that $v\alpha_s = \alpha_{s'}$ is the only element in $\operatorname{inv}(vsv^{-1}) = \operatorname{inv}(s')$.

Conversely, if $v\alpha_s = \alpha_{s'}$, the map vsv^{-1} only changes the $\alpha_{s'}$-coordinates of its input. It follows that each element in $\operatorname{inv}(vsv^{-1})$ must be a multiple of $\alpha_{s'}$. We saw in Theorem 26 that the only root in $\mathbb{R}_{\geqslant 0}\alpha_{s'}$ is $\alpha_{s'}$ itself, such that $\operatorname{inv}(vsv^{-1}) = \{\alpha_{s'}\}$ follows. In view of Theorem 26, we get $vsv^{-1} = s'$, finishing the proof. □

Exercise 31. Let $W = S_n$ be the symmetric group and $S = \{s_1, \ldots, s_{n-1}\}$ the standard transpositions.

(a) Show that the positive roots are given precisely by the roots of the form

$$\alpha_{i,j+1} = \alpha_{s_i} + \alpha_{s_{i+1}} + \cdots + \alpha_{s_{j-1}} + \alpha_{s_j}$$

for $1 \leqslant i \leqslant j \leqslant n-1$.

(b) Show that $\alpha_{i,j+1}$ lies in $\operatorname{inv}(w)$ if and only if $w(i) > w(j+1)$.

Exercise 32. Let $w = s_1 \cdots s_n$ be a reduced word. Define the *support* of w to be the smallest subset $S' \subseteq S$ such that $\operatorname{inv}(w)$ is contained in the subspace of V spanned by S'. Show that

$$\operatorname{supp}(w) = \{s_1, \ldots, s_n\}.$$

Exercise 33. Show that each $t \in T$ has a *symmetric reduced expression*, i.e. a reduced expression $t = s_1 \cdots s_n$ such that $s_{n+1-i} = s_i$ for $i = 1, \ldots, n$ (conversely, any symmetric expression in the above form will yield an element of T or 1).

6 Structure theory of Coxeter groups

Let W be a group and $S \subseteq W$ a subset such that S generates W, and each element in S has order 2 (in particular, $1 \notin S$). We define a length function $\ell : W \to \mathbb{Z}_{\geqslant 0}$ as in the case of Coxeter groups. Let $T \subset W$ denote the set of all elements in W which are conjugate to some element of S. The goal of this section is to introduce and halfway prove the following classification result.

Theorem 34. *The following are equivalent:*

(a) (W, S) *is a Coxeter group, i.e. there exists a function* $m : S \times S \to \mathbb{Z} \cup \{+\infty\}$ *such that* W *can be identified as the Coxeter group arising from the Coxeter system* (S, m).

(b) (W, S) *satisfies the* Strong Exchange Condition*: For each* $t \in T$ *and* $s_1, \ldots, s_n \in S$ *such that*

$$\ell(s_1 \cdots s_n t) \leqslant \ell(s_1 \cdots s_n),$$

we have

$$s_1 \cdots s_n t = s_1 \cdots s_{i-1} s_{i+1} \cdots s_n \text{ for some } i \in \{1, \ldots, n\}.$$

(c) (W, S) *satisfies the* Weak Exchange Condition*: For each reduced word* $w = s_1 \cdots s_n \in W$ *(i.e.* $s_1, \ldots, s_n \in S$ *and* $\ell(w) = n$*) and each* $s \in S$ *such that* $\ell(ws) \leqslant \ell(w)$*, we have*

$$ws = s_1 \cdots s_{i-1} s_{i+1} \cdots s_n \text{ for some } i \in \{1, \ldots, n\}.$$

(d) (W, S) *satisfies the* Deletion Condition*: For each non-reduced word $w = s_1 \cdots s_n \in W$ (i.e. $\ell(w) < n$), there exist indices $1 \leqslant i < j \leqslant n$ such that*

$$w = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_n.$$

We start by proving (a) $\implies$ (b).

Lemma 35. *Let (W, S) be a Coxeter group, $w = s_1 \cdots s_n$ a not necessarily reduced word and $\alpha \in \mathrm{inv}(w)$. Then there exists an index $i \in \{1, \ldots, n\}$ such that $\alpha = \sigma(s_n s_{n-1} \cdots s_{i+1})(\alpha_{s_i})$. For any such index, we have $ws_\alpha = s_1 \cdots s_{i-1} s_{i+1} \cdots s_n$ and $\ell(ws_\alpha) < \ell(w)$.*

Proof. Since $\alpha \geqslant 0$ and $w\alpha \leqslant 0$, we find an index i such that

$$\sigma(s_{i+1} \cdots s_n)(\alpha) \geqslant 0, \quad \sigma(s_i \cdots s_n)(\alpha) \leqslant 0.$$

In particular, $\sigma(s_{i+1} \cdots s_n)(\alpha)$ lies in $\mathrm{inv}(s_i)$, which we know to be $\{\alpha_{s_i}\}$. Thus $\alpha = \sigma(s_n \cdots s_{i+1})(\alpha_{s_i})$. By definition, it follows that

$$s_\alpha = s_n \cdots s_{i+1} s_i s_{i+1} \cdots s_n.$$

The claimed expression for ws_α is easily verified. The final length condition does not depend on the particular word $w = s_1 \cdots s_n$, so we may replace it by a reduced word, and then $ws_\alpha = s_1 \cdots \widehat{s_i} \cdots s_n$ is an even shorter word for ws_α. □

Certainly, the Strong Exchange Condition implies the Weak Exchange Condition.

Lemma 36. *It is always true that the Weak Exchange Condition is equivalent to the Deletion Condition.*

Proof. WEC $\implies$ DC: Assume that (W, S) satisfies the Weak Exchange Condition and that $w = s_1 \cdots s_n$ is a non-reduced word. This means that the function

$$f : \{1, \ldots, n\} \to \mathbb{Z}, \qquad j \mapsto \ell(s_1 \cdots s_j)$$

is not strictly increasing. In other words, we find an index such that $f(j) \geqslant f(j+1)$, or explicitly

$$\ell(s_1 \cdots s_j s_{j+1}) \leqslant \ell(s_1 \cdots s_j).$$

We choose j minimally with this property, then $s_1 \cdots s_j$ is reduced. Apply the Weak Exchange Condition to the reduced word $s_1 \cdots s_j$ and the element $s_{j+1} \in S$ to find the index i needed for the Deletion Condition.

DC $\implies$ WEC: If $s_1 \cdots s_n$ is reduced but $s_1 \cdots s_n s$ is not, apply the Deletion Condition to the non-reduced word $s_1 \cdots s_n s$. It is impossible to find indices $i < j \leqslant n$ such that

$$s_1 \cdots s_n s = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n s,$$

as this would contradict $s_1 \cdots s_n$ being reduced. Thus the Deletion Condition must yield an index i with

$$s_1 \cdots s_n s = s_1 \cdots \widehat{s_i} \cdots s_n,$$

as required for the Weak Exchange Condition. □

We see that a Coxeter group must satisfy both Exchange Conditions as well as the Deletion Condition. This is enough for the structure theory of Coxeter groups. The reverse implications however are still extremely useful, as they allow us to give examples of Coxeter groups.

Exercise 37. Using Theorem 34, prove the following result due to Deodhar [Deo89]:

Let (W, S) be a Coxeter group and let $W' \subseteq W$ be a subgroup generated by a set of reflections (i.e. a subset of T). Define

$$S' := \{t \in T \cap W' \mid \ell(tt') > \ell(t) \text{ for each } t' \in T \cap W' \backslash \{t\}\}.$$

Show that (W', S') is a Coxeter group.

Exercise 38. We might define the following Extra Weak Exchange Condition:

For each reduced word $w = s_1 \cdots s_n$ and each $s \in S$ such that $\ell(ws) < \ell(w)$, we have $ws = s_1 \cdots \widehat{s_i} \cdots s_n$ for some index i.

The only difference to the Weak Exchange Condition is that this version requires a strict inequality $\ell(ws) < \ell(w)$.

Give an example of a group W with a generating set of involutions S that is *not* a Coxeter group but satisfies this Extra Weak Exchange Condition (justifying why we did not use this extra weak version in Theorem 34)

7 Matsumoto's Theorem

In this section, we complete the proof of Theorem 34. Let (W, S) be a group together with a generating set of involutions. Define the function $m : S \times S \to \mathbb{Z} \cup \{+\infty\}$ by

$$m(s, s') = \text{order}(ss'),$$

i.e. $m(s, s')$ is the smallest positive integer such that $(ss')^{m(s,s')} = 1$ in W (or $+\infty$ if no such integer exists).

Definition 39. Let $(s_1, \ldots, s_n)$ be a word in S. We define the following transformations, yielding a new word in S:

(a) A *braid move* consists of replacing a subword of the form $(s, s', s, s', \ldots)$ with length $m(s, s') < +\infty$ by the word $(s', s, s', s, \ldots)$ of the same length.

(b) A *nil move* consists of replacing a subword of the form (s, s) by the empty word.

The following is an example of two braid moves followed by a nil move in the Coxeter group S_4.

$$\underline{s_1 s_2 s_1} s_3 s_2 s_3 \to s_2 s_1 \underline{s_2 s_3 s_2} s_3 \to s_2 s_1 s_3 s_2 \underline{s_3 s_3} \to s_2 s_1 s_3 s_2.$$

By definition of m and the fact that S consists of involutions, it follows that two words related by a sequence of braid and nil moves will represent the same element of W.

Theorem 40 ([Mat64])**.** *Suppose that (W, S) satisfies the Weak Exchange Condition. Let $(s_1, \dots, s_n)$ and $(s'_1, \dots, s'_\ell)$ be two words in S such that*

$$w := s_1 \cdots s_n = s'_1 \cdots s'_\ell \in W$$

and $\ell(w) = \ell$ (i.e. the second word is reduced, the first might not be). Then $(s_1, \dots, s_n)$ can be transformed into $(s'_1, \dots, s'_\ell)$ by a sequence of braid moves and nil moves.

Proof. Induction on $n \geqslant 0$. If $n = 0$, the first word is empty, thus $w = 1$ and we must have $\ell = 0$ as well. The claim is trivially satisfied.

Let us now assume that $n > 0$ and the corresponding claim has been proved for all words in S of length $< n$.

If $\ell(ws_n) > \ell(w)$, then $s_1 \cdots s_{n-1} = s'_1 \cdots s'_\ell s_n$ with the right-hand side reduced. By induction, we can transform $s_1 \cdots s_n$ into $s'_1 \cdots s'_\ell s_n s_n$ using a sequence of braid moves and nil moves (where we don't touch the rightmost letter). Apply another nil move to transform this into $s'_1 \cdots s'_\ell$.

Certainly, if $s_n = s'_\ell$, we get

$$s_1 \cdots s_{n-1} = s'_1 \cdots s'_{\ell-1}$$

and can apply induction immediately.

We may and will assume that $\ell(ws_n) \leqslant \ell(w)$ with $s_n \neq s'_\ell$. Applying the Weak Exchange Condition to $s'_1 \cdots s'_\ell$ and s_n shows that $\ell(ws_n) < \ell(w)$.

We consider *alternating words*, i.e. words in $\{s'_\ell, s_n\}$ of the form

$$(\dots, s'_\ell, s_n, s'_\ell, s_n, \dots)$$

of arbitrary length, without specifying whether the words starts/ends in s_n or s'_ℓ.

Pick such an alternating word $a = (\dots, s'_\ell, s_n, s'_\ell, s_n, \dots)$ of maximal length $m \geqslant 0$ such that

$$\ell(w\overline{a}) = \ell(w) - m,$$

here, $\overline{a} \in W$ denotes the element in W represented by the word a. Write the letters of a as $a = (a_1, \dots, a_m)$. Define a_{m+1} to be the unique element of $\{s_n, s'_\ell\}$ not equal to a_m.

Chose a reduced word ω for $w\overline{a}$, and denote by ωa the word obtained by composing ω with a. By choice of a, ωa is a reduced word for w.

We saw that $\ell(wa_{m+1}) < \ell(w)$, since $a_{m+1} \in \{s'_\ell, s_n\}$. Thus we may apply the Weak Exchange Property to the reduced word ωa and the simple reflection $a_{m+1} \in S$. We conclude that one of the following statements must hold true:

(1) wa_{m+1} is represented by a word of the form $\omega' a$ with ω' being obtained from ω by deleting one letter or

(2) wa_{m+1} is represented by a word of the form $\omega a'$ with a' being obtained from a by deleting one letter.

In case (1), we would get a reduced word for w given by $\omega' a a_{m+1}$, which contradicts the choice of m. So we must have (2).

Case (2) means that a is a reduced word but aa_{m+1} is not. Since $\ell(aa_{m+1}) \geqslant \ell(a) - 1$, the word a' must be reduced. It follows that a' is obtained from a by deleting the leftmost letter a_1 (as a' cannot contain a subword of the form ss, nor can a' end with a_{m+1}). We see that

$$\begin{aligned} & a_1 \cdots a_{m+1} = a_2 \cdots a_m \in W, \\ \text{or equivalently} \qquad & a_1 \cdots a_{m+1} a_m \cdots a_2 = 1. \end{aligned}$$

It follows that $m(s,s') \leqslant m$, in particular $m(s,s') < +\infty$. Moreover, $m \leqslant m(s,s')$ follows since a is reduced. Hence $m = m(s,s')$. Denote $\tilde{a} = (a_2, \dots, a_{m+1})$. Then we can perform a braid move to change between ωa and $\omega\tilde{a}$, both of which are reduced words for w.

One of these reduced words ends with s_n. If, say, ωa ends with s_n, then $\omega\tilde{a}$ ends with s'_ℓ and we can use induction to get

$$(s_1, \dots, s_n) \xrightarrow{\text{braid \& nil moves}} \omega a \xrightarrow{\text{braid move}} \omega\tilde{a} \xrightarrow{\text{braid moves}} (s'_1, \dots, s'_\ell).$$

If ωa ends with s'_ℓ, then $\omega\tilde{a}$ ends with s_n and we just have to interchange ωa with $\omega\tilde{a}$ in the above picture. □

From Matsumoto's theorem, we can immediately see the remaining direction (c) $\implies$ (a) of Theorem 34. The interested reader may take it as an exercise to formalize such a proof.

Remark 41. Associated with a Coxeter system (S, m), we can define the *Braid group* to be the group presented as follows: The generating set is given by S, and the relations are

$$\underbrace{(ss'ss'\cdots)}_{m(s,s')\text{ terms}} = \underbrace{(s'ss's\cdots)}_{m(s,s')\text{ terms}} \qquad \forall s, s' \in S \text{ s.th. } m(s,s') < +\infty.$$

There is a natural map from the Braid group to the associated Coxeter group, which is of course surjective but not injective. However, Matsumoto's theorem asserts that the *take a reduced word* map is well-defined from the Coxeter group to the Braid group.

Braid groups of type A_n have a nice geometric interpretation, where group elements can be understood as braids of $n+1$ strings and group multiplication being composition. Hence the name braid move. We might talk a bit more in depth about braid groups later, depending on time constraints and participants' interest.

Exercise 42. Given any word $w = s_1 \cdots s_n$, describe an algorithm to check if the word is reduced or not based on Matsumoto's theorem. Give a second algorithm that is based on Proposition 25.

Exercise 43. Using the results of section 1, prove that S_n is indeed a Coxeter group.

8 Parabolic subgroups

Consider a Coxeter group (W, S).

Definition 44. For each subset $I \subseteq S$, we define W_I to be the subgroup of W generated by I.

Subgroups of this form are known as *standard parabolic subgroups*. By *parabolic subgroups*, one understands subgroups equal or conjugate to standard parabolic subgroups, depending on the author.

Using Matsumoto's theorem, it is easy to see that $w \in W_I$ if and only if $\operatorname{supp}(w) \subseteq I$. It follows that $W_I \cap W_J = W_{I \cap J}$ for $I, J \subseteq S$.

Definition 45. For $L, R \subseteq S$, we define ${}^L W^R \subseteq W$ to be the subset of all those $w \in W$ that satisfy

$$\forall l \in L: \ \ell(lw) > \ell(w) \quad \text{and} \quad \forall r \in R: \ \ell(wr) > \ell(w).$$

If one of the sets is empty, we write

$$\begin{aligned} W^R &:= {}^{\varnothing}W^R = \{w \in W \mid \ell(ws) > \ell(w) \ \forall s \in R\}, \\ {}^L W &:= {}^L W^{\varnothing} = \{w \in W \mid \ell(sw) > \ell(w) \ \forall s \in L\}. \end{aligned}$$

Lemma 46. *Let $I \subseteq S$ and define $V_I \subseteq V$ to be the vector space spanned by the α_s for $s \in I$. Let $w \in W$. Then*

$$\begin{aligned} w \in W_I &\iff \operatorname{inv}(w) \subseteq V_I \\ w \in W^I &\iff \operatorname{inv}(w) \cap V_I = \varnothing. \end{aligned}$$

Proof. The first equivalence follows from the above remark on the support and Exercise 32.

If $w \notin W^I$, then $\alpha_s \in \operatorname{inv}(w)$ for some $s \in I$. Conversely, if $w \in W^I$, then $w\alpha_s \geqslant 0$ for all $s \in I$. Hence $w\alpha \geqslant 0$ for all $\alpha \in \Phi^+ \cap V_I$. □

Corollary 47. *For $I \subseteq S$, $w_1 \in W^I$ and $w_2 \in W_I$, we have*

$$\ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$

Proof. Indeed, observe that $w_2 \in W_I$ such that $\operatorname{inv}(w_1) \cap \operatorname{inv}(w_2^{-1}) = \varnothing$ follows. Conclude using Lemma 28. □

Proposition 48. *Let $L, R \subseteq S$ and $w \in W$.*

(a) The double coset

$$W_L w W_R = \{w_L w w_R \mid w_L \in W_L, w_R \in W_R\} \subseteq W$$

contains precisely one element of minimal length, denoted ${}^L w^R$. We have

$$(W_L w W_R) \cap {}^L W^R = \{{}^L w^R\}.$$

(b) There exist elements $w_L \in W_L$ and $w_R \in W_R$ such that

$$w = w_L\, {}^L w^R w_R \text{ and } \ell(w) = \ell(w_L) + \ell({}^L w^R) + \ell(w_R).$$

If $L = \varnothing$ or $R = \varnothing$, then the elements w_L, w_R are uniquely determined.

Proof. (a) Let $w_1 \in W_L w W_R$ be an element of minimal length. It is clear that such an element must lie in ${}^L W^R$.

Pick any element $w_2 \in (W_L w W_R) \cap {}^L W^R$. We need to prove that $w_1 = w_2$.

Since $w_2 \in W_L w_1 W_R$, we find elements $w_L \in W_L$ and $w_R \in W_R$ such that $w_2 = w_L w_1 w_R$.

For any such elements w_L, w_R, we apply Corollary 47 to see

$$\ell(w_2) + \ell(w_R) - \ell(w_L) = \ell(w_2 w_R^{-1}) - \ell(w_L) \leqslant \ell(w_L^{-1} w_2 w_R^{-1}) = \ell(w_1) \leqslant \ell(w_2).$$

Hence $\ell(w_R) \leqslant \ell(w_L)$. An analogous argument shows $\ell(w_L) \leqslant \ell(w_R)$, so that $\ell(w_L) = \ell(w_R)$.

Among all pairs $(w_L, w_R) \in W_L \times W_R$ with $w_2 = w_L w_1 w_R$, pick one such that $\ell(w_L) = \ell(w_R)$ gets minimal. If this length is zero, we see $w_1 = w_2$.

Otherwise, we find a simple reflection $s \in R$ such that $w_R \alpha_s \leqslant 0$. Since $w_1 \in W^R$, we see $w_1 w_R \alpha_s \leqslant 0$. However, $w_2 \in W^R$ again such that

$$w_L w_1 w_R \alpha_s = w_2 \alpha_s \geqslant 0 \implies \beta := -w_1 w_R \alpha_s \in \operatorname{inv}(w_L) \subseteq V_L.$$

We conclude

$$w_2 = w_L w_1 w_R = \underbrace{(w_L s_\beta)}_{\in W_L} w \underbrace{(w_R s)}_{\in W_R},$$

contradicting minimality of $\ell(w_R)$.

(b) Among all pairs $(w_L, w_R) \in W_L \times W_R$ with $w = w_L\, {}^L w^R w_R$, pick one such that $\ell(w_R)$ is as small as possible. We already know from Corollary 47 that

$$\ell(w_L\, {}^L w^R) = \ell(w_L) + \ell({}^L w^R).$$

In view of Lemma 28, it suffices to show that there exists no root $\alpha \in \Phi^+$ with $w_L\, {}^L w^R \alpha \leqslant 0$ and $w_R^{-1} \alpha \leqslant 0$.

Indeed, if α was such a root, then ${}^L w^R \alpha \geqslant 0$ as ${}^L w^R \in W^R$. Hence $\beta := {}^L w^R \alpha \in \operatorname{inv}(w_L) \subseteq V_L$ and we conclude

$$w = w_L\, {}^L w^R w_R = \underbrace{w_L s_\beta}_{\in W_L} \underbrace{{}^L w^R s_\alpha w_R}_{\in W_R},$$

contradicting minimality.

It remains to observe that if $L = \varnothing$ or $R = \varnothing$, we must have $w_L = 1$ resp. $w_R = 1$, and then the other element is uniquely determined by $w = w_L\, {}^L w^R w_R$. $\square$

Example 49. Consider the symmetric group $W = S_n$. Decompose the set $\{1, \ldots, n\}$ into blocks of adjacent numbers, i.e. subsets

$$B_k = \{b_k, b_k + 1, \ldots, b_{k+1} - 1\}$$

for numbers $1 = b_1 < \cdots < b_\ell = n + 1$. Call a permutation $w \in S_n$ *block-preserving* if $wB_k = B_k$ for all blocks B_k.

Let $I \subseteq \{s_1, \ldots, s_{n-1}\}$ be the subset of those standard transpositions $s_i = (i\, i{+}1)$ such that i and $i+1$ lie in the same block. Then W_I is precisely the group of block-preserving permutations. Note that each standard parabolic subgroup of S_n arises in this way.

The set W^I consists of all permutations $w \in W$ which are *block monotonic*, i.e. satisfy $w(i) < w(j)$ whenever $i < j$ lie in the same block.

Exercise 50. Let $w \in S_5$ be the permutation $w = (1\ 2\ 3)(4\ 5)$.

(a) Determine the length and the support of w.

(b) Let $L = \{s_2, s_3\}$ and $R = \{s_2, s_3, s_4\}$. Compute ${}^L w^R$ and provide elements w_L, w_R as in Proposition 48 (b).

Exercise 51. Let $R_1, R_2 \subseteq S$. Show that $W^{R_1 \cup R_2} = W^{R_1} \cap W^{R_2}$.

Exercise 52. Give an example of a Coxeter group (W, S), an element $w \in W$ and subsets $L, R \subseteq S$ such that the elements w_L, w_R as in Proposition 48 (b) are not unique.

Exercise 53. In the setting of a general Coxeter group (W, S) with $w \in W$ and $L, R \subseteq S$, prove that

$${}^L w^R = {}^L(w^R) := {}^L({}^\varnothing w^R)^\varnothing.$$

Exercise 54. Let $L, R \subseteq S$ and $w \in {}^L W^R$. Let $R_1 \subseteq R$ be the subset of those reflections $s \in R$ with $wsw^{-1} \in L$. Show that there is a bijective map

$$(W_L \times W_R)\big/\!\sim\ \to W_L w W_R,$$

where we identify the pairs (w_L, sw_R) and $(w_L wsw^{-1}, w_R)$ for $w_L \in W_L, w_R \in W_R$ and $s \in R_1$.

Exercise 55. Let $L, R \subseteq S$ and $w \in W$. Show that $\ell({}^L w^R)$ is given by the number of roots $\alpha \in \Phi^+ \backslash V_R$ such that $w\alpha \in \Phi^- \backslash V_L$.

9 Bruhat order

Let (W, S) be a Coxeter group.

Definition 56. If $w \in W$ and $t \in T$ are such that $\ell(w) < \ell(wt)$, we write $w \to wt$. We say that $w \leqslant w'$ in the Bruhat order if there is a sequence of such arrows

$$w = w_1 \to w_2 \to \cdots \to w_m = w'.$$

We note the following immediate properties.

Lemma 57. *Let $v, w \in W$ and $t \in T$.*

(a) If $v < w$ then $\ell(v) < \ell(w)$.

(b) The Bruhat order is a partial order.

(c) We have $1 \leqslant w$.

(d) $w < tw$ if and only if $\ell(w) < \ell(tw)$.

(e) $v \leqslant w$ if and only if $v^{-1} \leqslant w^{-1}$. □

Example 58. Let G be a Lie group, $B \subset G$ a Borel subgroup and W the Weyl group. We know that $G = \bigsqcup_{w \in W} BwB$.

The group G carries the structure of a topological space, so we can consider the closure of a double coset $\overline{BwB}$. This is naturally a union of double cosets again, so we may ask which double cosets $Bw'B$ appear as subsets of $\overline{BwB}$. It turns out that this is the case if and only if $w' \leqslant w$ in the Bruhat order. In other words,

$$\overline{BwB} = \bigsqcup_{w' \leqslant w} BwB.$$

We give equivalent descriptions of this order.

Definition 59. A *Bruhat cover* is a pair $(w, w') \in W \times W$ such that $w < w'$ and $\ell(w') = \ell(w) + 1$. We write $w \lessdot w'$.

Note that if (w, w') is a Bruhat cover, then $w^{-1}w' \in T$.

Theorem 60. *Let $w = s_1 \cdots s_{\ell(w)} \in W$ be a reduced word and $v \in W$. The following are equivalent.*

(a) $v \leqslant w$ in the Bruhat order.

(b) There are indices $1 \leqslant i_1 < \cdots < i_k \leqslant \ell(w)$ such that $v = s_{i_1} \cdots s_{i_k}$.

(c) There are indices $1 \leqslant i_1 < \cdots < i_k \leqslant \ell(w)$ such that $v = s_{i_1} \cdots s_{i_k}$ and $k = \ell(v)$.

(d) There is a sequence of Bruhat covers

$$v = v_0 \lessdot v_1 \lessdot \cdots \lessdot v_{\ell(w)-\ell(v)} = w.$$

Proof. (a) $\implies$ (b): Take a sequence

$$w = w_1 \to \cdots \to w_m = v$$

as in the definition of the Bruhat order and apply the Strong Exchange Property $m - 1$ times.

(b) $\implies$ (c): Any word representing $v \in W$ contains a reduced subword representing v, this follows from the Deletion Property.

(c) $\implies$ (d): Induction on $\ell(w) - \ell(v)$. If $\ell(v) = \ell(w)$, there is nothing to show.

Otherwise, we can denote by $h \in \{1, \ldots, \ell(w)\}$ the smallest value not equal to any of the $i_1, \ldots, i_k$. Among all reduced subwords representing v, choose one such that this $h \in \{1, \ldots, \ell(w)\}$ is as large as possible. We let $v' \in W$ be the element obtained by inserting s_i into this reduced word for v, i.e.

$$v' = s_1 \cdots s_{i-1} s_h s_{i_h} \cdots s_{i_k}$$

(using the fact $i_j = j$ for $j < h$). Let $\alpha := s_{i_k} \cdots s_{i_h}(\alpha_{s_h}) \in \Phi$, where α_{s_h} denotes the simple root associated with the simple reflection s_h.

If $\alpha \in \Phi^-$, we can apply the Weak Exchange Property to write

$$s_i s_{i_h} \cdots s_{i_k} = s_{i_h} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_k}$$

for some $j \in \{h, \ldots, k\}$. Then

$$v = s_1 \cdots s_h s_{i_h} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_{\ell(w)}}$$

is a reduced word representing v starting with $s_1 \cdots s_h$, contradicting the choice of h.

If $\alpha \in \Phi^+$, we see that

$$v\alpha = s_1 \cdots s_{h-1}(\alpha_{s_h}) \in \Phi^+$$

as $s_1 \cdots s_{\ell(w)}$ is reduced. Hence $\ell(v') = \ell(vs_\alpha) > \ell(v)$. In particular, the above word for v' must be reduced and $v \lessdot v'$. We can apply induction to finish the proof.

The final implication (d) $\implies$ (a) is trivial. $\square$

Corollary 61. *Let $v, w \in W$ and $s \in S$ such that $ws < w$.*

(a) If $vs < v$, then $v \leqslant w$ if and only if $vs \leqslant ws$.

(b) If $vs > v$, then $v \leqslant w$ if and only if $v \leqslant ws$.

Proof. Pick a reduced word of $w = s_1 \cdots s_m$ ending with s and apply the subword criterion Theorem 60 (c). Then part (b) as well as the implication "$\impliedby$" in (a) follow immediately.

It remains to show the implication "$\implies$" in (a). If $v \leqslant w$, then $vs \leqslant v \leqslant w$. Apply (b) to get $vs \leqslant ws$. $\square$

Exercise 62. Let $s \in S$ and $w \in W$. Show that $s \leqslant w$ if and only if $s \in \operatorname{supp}(w)$.

Exercise 63. Give an example of a Coxeter group (W, S), two elements $v \leqslant w$ in W and a reduced word

$$v = s_1 \cdots s_{\ell(v)}$$

that is *not* a subword of any reduced word for w.

Exercise 64. Show that if (W_I, I) is a standard parabolic subgroup of (W, S), then the Bruhat order on W_I is the restriction of the Bruhat order on W.

Exercise 65. Show that for any finite subset $F \subseteq W$, there exists some $w \in W$ with $v \leqslant w$ for all $v \in F$.

10 Bruhat order on parabolic quotients

Let (W, S) be a Coxeter group and $L, R \subseteq S$ be two arbitrary subsets.

Lemma 66. *Let $v \in {}^LW^R$ and $w \in W$. Then*

$$v \leqslant w \iff v \leqslant {}^Lw^R.$$

Proof. Decompose $w = w_L\, {}^Lw^R w_R$ as in Proposition 48 (b). Choose reduced words for w_L, ${}^Lw^R$ and w_R and concatenate them to a reduced word for w.

Since $v \in {}^LW^R$, no reduced word for v can start with an element of L nor end with an element of R. Applying the subword criterion from Theorem 60 (c), we conclude. $\square$

Corollary 67. *If $v \lessdot w$ with $v \in {}^LW^R$, then $w \in {}^LW^R$ or $w = sv$ for some $s \in L$ or $w = vs$ for some $s \in R$.*

Proof. $v \leqslant {}^Lw^R \leqslant w$. One of them must be an equality. $\square$

Proposition 68. *If $v, w \in W^R$, then $v \leqslant w$ if and only if there exists a chain of Bruhat covers*

$$v = v_0 \lessdot v_1 \lessdot \cdots \lessdot v_{\ell(w)-\ell(v)} = w$$

consisting of elements in W^R.

Proof. Induction on $\ell(w)$. If $w = 1$, there is nothing to show.

Pick a simple reflection s with $sw < w$. Then $sw \in W^R$.

If $sv > v$, we get $v \leqslant sw \lessdot w$ by Corollary 61 and are done.

We may hence assume that $v > sv \in W^R$. By induction, there exists some $v' \in W^R$ with $sv \leqslant v' \lessdot sw$.

If $sv' < v'$, we see $v \leqslant v' \lessdot sw \lessdot w$ and are done by induction.

If $sv' > v'$, then $v \leqslant sv' \lessdot w$. In case $sv' \in W^R$, we are again done by induction. If however $sv' \notin W^R$, Corollary 67 shows $v' = (sv')^R$. Hence $v \leqslant v' \lessdot sv' \lessdot w$. We are again done by induction. $\square$

The main result of this section is the following result, known as *Deodhar's lemma.*

Lemma 69 ([Deo77]). *Let $\{R_r \subseteq S\}_{r\in\rho}$ be a family of subsets of S and $R = \bigcap_{r\in\rho} R_r$ its intersection.*

For $v, w \in W$, we have $v^R \leqslant w$ if and only if $v^{R_r} \leqslant w$ for each $r \in \rho$.

Proof. Induction on $\ell(w)$. If $\ell(w) = 0$, we get $v^{R_r} = 1$ so $v \in W_{R_r}$ for each $r \in \rho$. In particular, $v \in W_R$, so that $v^R = 1$.

Let now $\ell(w) > 0$ and pick a simple reflection s with $sw < w$. Let $L = \{s\}$.

By Corollary 61, we have $v^R \leqslant w$ if and only if ${}^Lv^R \leqslant sw$ (using Exercise 53). Similarly, $v^{R_r} \leqslant w$ if and only if ${}^Lv^{R_r} \leqslant sw$. We summarize:

$$v^R \leqslant w \iff {}^Lv^R \leqslant sw \overset{\text{ind.}}{\iff} \forall r: \ {}^Lv^{R_r} \leqslant sw \iff \forall r: \ v^{R_r} \leqslant w.$$

This finishes the induction and the proof. $\square$

The two-sided analogue of Deodhar's lemma holds true as well.

Theorem 70. *Let λ, ρ be arbitrary indexing sets and pick subsets*

$$\{L_\ell \subseteq S\}_{\ell \in \lambda}, \qquad \{R_r \subseteq S\}_{r \in \rho}.$$

Define

$$L := \bigcap_{\ell \in \lambda} L_\ell, \qquad R := \bigcap_{r \in \rho} R_r.$$

For $v, w \in W$, we have ${}^L v^R \leqslant w$ if and only if for each $(\ell, r) \in \lambda \times \rho$, we have

$${}^{L_\ell} v^{R_r} \leqslant w.$$

Proof. The direction "$\implies$" is trivial since ${}^{L_\ell} v^{R_r} \leqslant {}^L v^R$.

Suppose now that ${}^{L_\ell} v^{R_r} \leqslant w$ for all ℓ, r. In view of Lemma 69, it suffices to show ${}^L v^{R_r} \leqslant w$ for all $r \in \rho$. So fix $r \in \rho$ for the remainder of the proof.

Apply Lemma 69 to the sets L_ℓ and the elements

$$\left(v^{R_r}\right)^{-1}, \quad w^{-1} \in W$$

to see that

$${}^L v^{R_r} \leqslant w \iff \forall \ell : \ {}^{L_\ell} v^{R_r} \leqslant w.$$

This finishes the proof. □

A typical, and arguably optimal, choice for given v, w is to let

$$\begin{aligned} &\lambda = \{s \in S \mid sv < v\}, \quad \rho = \{s \in S \mid vs < v\} \\ &L_\ell = S \backslash \{\ell\} \text{ and } R_r = S \backslash \{r\}. \end{aligned}$$

Then ${}^L v^R = v$. The elements ${}^{L_\ell} v^{R_r}$ are usually much smaller than v, and come with a bit of extra structure (i.e. all reduced words starting and ending with the same fixed simple reflections).

Exercise 71. Let $L, R \subseteq S$ and $v \leqslant w$. Show that ${}^L v^R \leqslant {}^L w^R$.

Exercise 72. Let $v, w \in {}^L W^R$ and $v \leqslant w$. Show that the following are equivalent:

- For each $u \in W$ with $v \leqslant u \leqslant w$, we have $u \in {}^L W^R$.
- For each $s \in L$, we have $sv \nleqslant w$ and for each $s \in R$, we have $vs \nleqslant w$.

Exercise 73. Give an example of a Coxeter group (W, S), subsets $L, R \subseteq S$ and two elements $v, w \in {}^L W^R$ such that $v < w$, $\ell(w) - \ell(v) \geqslant 2$ and there is no $u \in {}^L W^R$ with $v < u < w$ (explaining why we can prove Proposition 68 only for one-sided quotients).

11 The Tableau criterion

In this section, we focus on the example of the symmetric group $W = S_n$. Recall that the set of simple reflections is given by the standard transpositions $s_i = (i\ i+1)$. The set of reflections is $T = \{(i\ j) \mid i \neq j\}$. The set of roots can be identified with

$$\Phi = \{e_i - e_j \mid i \neq j\} \subseteq V = \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_1 + \cdots + v_n = 0\}.$$

The positive roots are those $e_i - e_j$ where $i < j$. It follows $w < w(i\ j)$ in the Bruhat order if and only if $w(i) < w(j)$ for $i < j$.

Lemma 74. *Let $i, j \in \{1, \ldots, n-1\}$ and $R := S \backslash \{s_i\}, L := S \backslash \{s_j\}$. For*

$$\max(1, i+j-n+1) \leqslant k \leqslant \min(i,j) + 1,$$

define $w_k \in W$ to be the permutation

$$w_k : \{1, \ldots, n\} \to \{1, \ldots, n\},\ h \mapsto \begin{cases} h, & h < k, \\ h + j - k + 1, & k \leqslant h \leqslant i, \\ h + k - i - 1, & i + 1 \leqslant h \leqslant i + j - k + 1, \\ h, & h > i + j - k + 1. \end{cases}$$

So the permutation matrix of w is given by the block matrix

$$w|_V : \begin{pmatrix} \mathrm{Id}_{k-1} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{j-k+1} & 0 \\ 0 & \mathrm{Id}_{i-k+1} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-i-j+k-1}. \end{pmatrix}$$

(a) The set ${}^L W^R$ is given by

$${}^L W^R = \{w_k \mid \max(1, i+j-n+1) \leqslant k \leqslant \min(i,j) + 1\}.$$

(b) For $w \in W$, let $k := 1 + \#\{h \leqslant i \mid w(h) \leqslant j\}$. Then ${}^L w^R = w_k$.

(c) If both w_k and w_{k+1} are defined, then $w_k > w_{k+1}$ in the Bruhat order.

Proof. (a) The claim $w_k \in {}^L W^R$ is easily verified, we have to show that these exhaust this set. Note that $w_k = 1$ if $k = \min(i,j) + 1$.

Let $w \in {}^L W^R \backslash \{1\}$. This means that whenever $h_1 < h_2$ with $w(h_1) > w(h_2)$, we must have $h_1 \leqslant i < h_2$ and $w(h_1) > j \geqslant w(h_2)$.

Since $w \neq 1$, we let $k \in \{1, \ldots, n\}$ be the smallest integer such that $w(k) \neq k$. From $w(\{1, \ldots, k-1\}) = \{1, \ldots, k-1\}$, we get $w(k), w^{-1}(k) > k$. By the above considerations, it follows that $k \leqslant i < w^{-1}(k)$ and $w(k) > j \geqslant k$.

Since $k < w(k) < \cdots < w(i)$, we must have $w^{-1}(h) > i$ for $k \leqslant h < w(k)$. Note that $w(i+1) < \cdots < w(n)$, so that

$$w(i+1) = k, w(i+2) = k+1, \ldots, w(i + w(k) - k) = w(k) - 1.$$

Since $k < w^{-1}(k) < \cdots < w^{-1}(j)$, we must have $w(h) > j$ for $k \leqslant h < w^{-1}(k)$. Note that $w^{-1}(j+1) < \cdots < w^{-1}(n)$, so that

$$w^{-1}(j+1) = k, w^{-1}(j+2) = k+1, \ldots, w^{-1}(j + w^{-1}(k) - k) = w^{-1}(k) - 1.$$

Substituting $w(k) = j+1$ and $w^{-1}(k) = i+1$, we see that $w(h) = w_k(h)$ for $h \leqslant i+j-k+1$. It follows that $w(\{1, \ldots, i+j-k+1\}) = \{1, \ldots, i+j-k+1\}$. Thus $w(h) > i+j-k+1$ for $h > i+j-k+1$. Since $j+1 \geqslant k$, we get

$$w(i+j-k+2) < \cdots < w(n),$$

so that $w = w_k$ follows.

(b) The cardinality in question does not change when multiplying by a simple reflection in L on the left, or a simple reflection in R on the right. By (a), it suffices to show the claim for the w_k, which is easily verified.

(c) Suppose that

$$\max(1, i+j-n+1) \leqslant k \leqslant \min(i,j).$$

Then $w_{k+1}(k) = k$ and $w_{k+1}(n) = n$. Thus $w_{k+1} < w$, where $w = w_{k+1}(k\ n)$ is the product of w_{k+1} with the reflection $(k\ n)$. By (b), we have ${}^{L}w^{R} = w_k$. By Lemma 66, we get $w_{k+1} \leqslant w_k$. $\square$

Definition 75. Let $w \in W$ and $i, j \in \{1, \ldots, n\}$. We define

$$w[i,j] = \#\{h \leqslant i \mid w(h) > j\}.$$

Theorem 76 (Tableau Criterion)**.** *Let $v, w \in W$. Then the following are equivalent:*

(a) $v \leqslant w$ in the Bruhat order.

(b) For all $i, j \in \{1, \ldots, n\}$, we have $v[i,j] \leqslant w[i,j]$.

(c) For all $i \in \{2, \ldots, n\}$ and $j \in \{1, \ldots, n-1\}$ such that

$$v^{-1}(i) > v^{-1}(i+1) \text{ and } v(j) > v(j+1),$$

we have $v[i,j] \leqslant w[i,j]$.

Proof. Since $v[1, \bullet] = 0 = v[\bullet, n]$, we may restrict our attention to $i > 1$ and $j < n$ in (b). By the previous lemma, the condition $v[i,j] \leqslant w[i,j]$ is equivalent to ${}^{L}v^{R} \leqslant {}^{L}w^{R}$ for $R = S \backslash \{s_{i-1}\}, L = S \backslash \{s_j\}$. Now (a) $\Longrightarrow$ (b) is Exercise 71. The implication (c) $\Longrightarrow$ (a) is the two-sided version of Deodhar's lemma, Theorem 70. $\square$

Example 77. Consider the group $W = S_4$ with the permutations given as follows:

$$v = (1\ 3\ 2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad w = (1\ 4\ 3) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Inspecting the element v, we only have to check $v[i,j] \leqslant w[i,j]$ for $i = j = 2$. We compute $v[2,2] = 1$ and $w[2,2] = 1$. Thus $v \leqslant w$.

Exercise 78. Show that $w_k > w_{k+1}$ by exhibiting a sequence of Bruhat arrows $w_k \to \cdots \to w_{k+1}$.

Exercise 79. Let $v, w \in W$ and $i \in \{1, \ldots, n\}$. Let $v_i \in \mathbb{Z}^i$ be the vector obtained by sorting the sequence $v(1), \ldots, v(i)$ so that it becomes increasing, and similarly w_i. Show that the following are equivalent:

- $v[i,j] \leqslant w[i,j]$ for all $j \in \{1, \ldots, n\}$.
- We have $v_i \leqslant w_i$ component-wise in $\mathbb{Z}^i$.

Conclude that $v \leqslant w$ if and only if $v_i \leqslant w_i$ whenever $v(i) > v(i+1)$.

Arranging the corresponding vectors $v_\bullet$ and $w_\bullet$ nicely, one obtains so-called "tableaux", giving the name of the Tableau Criterion. These structures and their relationship to the combinatorics and representation theory of the symmetric group are beyond the scope of this course (unless the audience insists), so we refer to [BB05, Appendix A3], as well as any treatment of the representation theory of symmetric groups (Wikipedia is a good starting point).

12 Finite Coxeter groups

Let (W, S) be a Coxeter group.

Proposition 80. *Let $w_0 \in W$. Then*

$$(\forall w \in W:\ \ell(w) \leqslant \ell(w_0)) \iff (\forall s \in S:\ w_0 s < w_0).$$

There exists an element w_0 satisfying these properties if and only if W is finite. If such w_0 exists, it is uniquely determined and satisfies moreover the following properties:

(a) $w_0^2 = 1$.

(b) $\ell(w_0) = \#T = \#\Phi^+$.

(c) For any $w \in W$, we have $\ell(ww_0) = \ell(w_0 w) = \ell(w_0) - \ell(w)$.

(d) For any $s \in S$, we have $w_0 s w_0 \in S$ and $w_0(\alpha_s) = -\alpha_{w_0 s w_0}$.

(e) Any element $w \in W$ satisfies $w \leqslant w_0$.

Proof. If $w_0 \in W$ has maximal length, then $\ell(w_0 s) \leqslant \ell(w_0)$ for all $s \in S$, showing the implication " $\implies$ ".

If $w_0 s < w_0$ for all $s \in S$, we see $w_0(\alpha) \leqslant 0$ for all $\alpha \in \Phi^+$. Thus $\mathrm{inv}(w_0) = \Phi^+$.

Such an element is uniquely determined and satisfies $\mathrm{inv}(w) \subseteq \mathrm{inv}(w_0)$ for all $w \in W$. In particular, we get the implication " $\impliedby$ ".

If W_0 is finite, then there trivially exists an element of maximal length. If w_0 exists, then each $w \in W$ is uniquely determined by $\mathrm{inv}(w) \subseteq \mathrm{inv}(w_0)$, and there are only finitely many possibilities for those subsets.

Suppose now that W is finite and w_0 satisfies the defining properties.

(a) Since w_0^{-1} has the same length as w_0, it must be equal to w_0.

(b) We already calculated $\mathrm{inv}(w_0) = \Phi^+$.

(c) The first identity follows from Lemma 28 together with the above observation $\mathrm{inv}(w^{-1}) \subseteq \mathrm{inv}(w_0)$.

The other equation follows from

$$\ell(ww_0) = \ell((ww_0)^{-1}) = \ell(w_0 w^{-1}) = \ell(w_0) - \ell(w^{-1}) = \ell(w_0) - \ell(w).$$

(d) Note that

$$\begin{aligned}\ell(w_0 s w_0) =&\ell(w_0) - \ell(s w_0) = \ell(w_0) - \ell((s w_0)^{-1}) \\ =&\ell(w_0) - \ell(w_0 s) = \ell(s) = 1.\end{aligned}$$

Thus $\mathrm{inv}(w_0 s w_0) = \{\alpha_{w_0 s w_0}\}$, proving $w_0 \alpha_s = \pm \alpha_{w_0 s w_0}$. We already saw $w_0 \alpha_s \leqslant 0$.

(e) This is an easy consequence of (c): Concatenate a reduced word for w with a reduced word for $w_0 w$ to obtain a reduced word for w_0 containing that reduced word for w as a subword. ☐

Example 81. In the finite Coxeter group $W = S_n$, the longest element is given by $w_0(i) = n + 1 - i$.

Definition 82. Let $L, R \subseteq S$ be subsets and define

$${}^{-L}W^{-R} := \{w \in W \mid sw < w \text{ for all } s \in L \text{ and } ws < w \text{ for all } s \in R\}.$$

We say that L is *spherical* if W_L is finite.

Lemma 83. *Let $L, R \subseteq S$ and $w \in W$.*

(a) The intersection $(W_L w W_R) \cap {}^{-L}W^{-R}$ is non-empty if and only if both W_L and W_R are spherical.

If both W_L and W_R are spherical, the double coset $W_L w W_R$ contains a unique element of maximal length, denoted ${}^{-L}w^{-R}$. Moreover,

$$(W_L w W_R) \cap {}^{-L}W^{-R} = \{{}^{-L}w^{-R}\}.$$

(b) *Suppose that L, R are spherical. Then there are elements $w_L \in W_L, w_R \in W_R$ such that ${}^{-L}w^{-R} = w_L w w_R$ is a length-additive product:*

$$\ell({}^{-L}w^{-R}) = \ell(w_L) + \ell(w) + \ell(w_R).$$

(c) *Suppose that L, R are spherical. For $v \in W$, we have*

$${}^{L}v^{R} \leqslant w \iff {}^{L}v^{R} \leqslant {}^{-L}w^{-R} \iff v \leqslant {}^{-L}w^{-R}.$$

Proof. Let us first consider the case that L is not spherical. Then $W^{-L} = \varnothing$: Indeed, if $v \in W^{-L}$, we could write $v = v_1 v_2$ with $v_1 \in W^L$ and $v_2 \in W_L$. Thus $v_2\alpha_s \leqslant 0$ for all $s \in L$, showing that L is spherical. The same argument works for R being spherical.

Assume for the remainder of the proof that both W_L and W_R are spherical. The arguments are very analogous to the proof of Proposition 48, so we leave them as an exercise to the interested reader (we won't use these results). $\square$

Exercise 84. Let R be spherical. Show that the elements of W^{-R} are precisely those of the form $ww_0(R)$ where $w \in W^R$ and $w_0(R) \in W_R$ is the longest element.

Exercise 85. Show that for finite Coxeter groups of type B_n for $n \geqslant 3$, we have $w_0 s w_0 = s$ for all $s \in S$.

Exercise 86. Let (W, S) be a finite Coxeter group and $v, w \in W$. Show that $v \leqslant w$ if and only if $w_0 v \geqslant w_0 w$.

13 Weak Order

Let (W, S) be a Coxeter group.

Definition 87. Let $v, w \in W$. We say that v is less than w in the *left weak order* and write $v \leqslant_L w$ if $\ell(w) = \ell(wv^{-1}) + \ell(v)$.

We say that v is less than w in the *right weak order* and write $v \leqslant_R w$ if $\ell(w) = \ell(v) + \ell(v^{-1}w)$.

One has to verify that these define partial orders, but this is not too hard. So $v \leqslant_R w$ means that there is some reduced word for w that *starts* with a reduced word for v. The inequality $v \leqslant_L w$ implies $v \leqslant w$, hence the name "weak order". In a finite Coxeter group, the longest element is the unique maximum with respect to both weak orders.

The concepts of left and right weak order are closely related, e.g. by passing to inverses. In any case, they ask which pairs $(w_1, w_2) \in W$ satisfy the length additivity condition $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. If G is a Lie group with Borel B and Weyl group W, this length additivity condition is equivalent to the geometric condition that the product of Borel double cosets / Schubert cells $Bw_1B \cdot Bw_2B$ should be a Borel double coset again, namely Bw_1w_2B.

We get the following characterization analogous to Bruhat order.

Lemma 88. *For $v, w \in W$, the following are equivalent* *(a)* $v \leqslant_R w$.

(b) *There is a sequence*

$$v = v_1 \xrightarrow{t_1} v_2 \xrightarrow{t_2} \cdots \xrightarrow{t_n} v_n = w$$

as in the definition of Bruhat order, with the additional restriction that $t_i \in S$ for all i.

(c) *There is a sequence of right weak order covers*

$$v = v_1 \lessdot_R v_2 \lessdot_R \cdots \lessdot_R v_n = w$$

with $v_i \lessdot_R v_{i+1}$ means $v_i \leqslant_R v_{i+1}$ and $\ell(v_{i+1}) - \ell(v_i) = 1$.

(d) *We have* $\mathrm{inv}(v^{-1}) \subseteq \mathrm{inv}(w^{-1})$.

Proof. (a) $\implies$ (b): Just pick a reduced word for $v^{-1}w$ to obtain the required t_i.

(b) $\implies$ (c): Trivial.

(c) $\implies$ (d): We may assume $n = 2$, i.e. $w = vs$ for a simple reflection $s \in S$. Then Proposition 25 shows $\mathrm{inv}(w^{-1}) = \mathrm{inv}(v^{-1}) \cup \{v\alpha_s\}$.

(d) $\implies$ (a): It follows for each root $\alpha \in \Phi^-$ with $v\alpha \in \Phi^+$ that $w^{-1}v\alpha \in \Phi^-$. By Lemma 28, $\ell(w) = \ell(v) + \ell(v^{-1}w)$. □

Of course, we get an analogous lemma for the left weak order up to passing to inverses.

Lemma 89. *For $v, w \in W$, the following are equivalent*

(a) $v \leqslant_L w$.

(b) *There is a sequence*

$$v = v_1 \xrightarrow{t_1} v_2 \xrightarrow{t_2} \cdots \xrightarrow{t_n} v_n = w$$

as in the definition of Bruhat order, with the additional restriction that $v_i t_i v_i^{-1} \in S$ for all i.

(c) *There is a sequence of left weak order covers*

$$v = v_1 \lessdot_L v_2 \lessdot_L \cdots \lessdot_L v_n = w$$

with $v_i \lessdot_L v_{i+1}$ means $v_i \leqslant_L v_{i+1}$ and $\ell(v_{i+1}) - \ell(v_i) = 1$.

(d) *We have* $\mathrm{inv}(v) \subseteq \mathrm{inv}(w)$. □

Proposition 90. *Let $E \subseteq W$ be an non-empty set of elements of W. Then an* infimum *E exists, i.e. the set of lower bounds*

$$\mathrm{lb}(E) = \{v \in W \mid v \leqslant_L w \text{ for all } w \in E\}$$

contains a unique maximum with respect to $\leqslant_L$.

Proof. Induction on $\min\{\ell(w) \mid w \in E\}$. If $1 \in E$, then 1 is the only lower bound of E.

In the inductive step, let

$$I := \{s \in S \mid ws < w \text{ for all } w \in E\}.$$

If $v \in \mathrm{lb}(E)$ and $vs < s$, then $s \in I$. So if $I = \varnothing$, we get $\mathrm{lb}(E) = \{1\}$ and are done again.

Otherwise, pick $s \in I$ define $E' := \{ws \mid w \in W\}$. By induction, there is an infimum z of E'. Then zs is an infimum of E. $\square$

Exercise 91. Show that the infimum of a set is uniquely determined. Show that if $E \subseteq W$ is a non-empty set with an upper bound, then a supremum (= smallest upper bound) exists.

Exercise 92. Let $W = S_5$, $v = (1\ 3\ 2\ 4\ 5)$ and $w = (1\ 4\ 2\ 5\ 3)$. Show that $v \not\leqslant_L w$ and $v \not\leqslant_R w$.

Exercise 93. Let $v, w \in W$ and $z \in W$ be the infimum of $\{v, w\}$ with respect to $\leqslant_L$. Show that $\mathrm{inv}(z) \subseteq \mathrm{inv}(v) \cap \mathrm{inv}(w)$ and give an example where the inclusion is strict.

14 Foldings

In this section, we introduce an important concept to create new Coxeter groups from old ones, as well as understand automorphisms of Coxeter systems.

Let (W, S) be a Coxeter group. Let Γ be a group that acts on S such that

$$m(s, s') = m(\gamma.s, \gamma.s') \text{ for all } s, s' \in S \text{ and } \gamma \in \Gamma.$$

Then each $\gamma \in \Gamma$ induces a group automorphism of W that preserves S (i.e. a *Coxeter group automorphism*). We denote by W^Γ the set of those elements $w \in W$ with $\gamma.w = w$ for all $\gamma \in \Gamma$.

Lemma 94. *Let $w \in W^\Gamma$ and $I := \{s \in S \mid ws < w\}$. Then I is a spherical subset of S that is preserved by all $\gamma \in \Gamma$. Denoting by $w_0(I) \in W_I$ the longest element, we have*

$$w = w^I w_0(I), \qquad w^I \in (W^I) \cap (W^\Gamma).$$

Proof. Since $w \in W^{-I}$, we saw already that I must be spherical. The condition $\gamma.w = w$ implies $\gamma(I) = I$ for all $\gamma \in \Gamma$.

We know that we can always write w as a length additive product $w = w^I w_I$ for some $w^I \in W^I$ and $w_I \in W_I$. By choice of I, we have $w_I = w_0(I)$. The condition $\gamma(I) = I$ implies $\gamma.w_0(I) = w_0(I)$, hence $w_0(I) \in W^\Gamma$. We conclude $w^I \in W^\Gamma$. $\square$

Definition 95. (a) We say that $I \subseteq S$ is a *spherical* Γ-*orbit* if I spherical and of the form $I = \{\gamma.s \mid \gamma \in \Gamma\}$ for some $s \in S$.

(b) For $I \subseteq S$ a spherical Γ-orbit, define

$$\alpha_I := \sum_{s \in I} \alpha_s \in V^\Gamma := \{v \in V \mid \gamma.v = v \text{ for all } \gamma \in \Gamma\}.$$

(c) Define $S^\Gamma \subseteq W^\Gamma$ to be the set

$$S^\Gamma = \{w_0(I) \mid I \subseteq S \text{ is a spherical } \Gamma\text{-orbit.}\}.$$

Observe that the α_I for spherical Γ-orbits I form a basis of V^Γ.

Lemma 96. *For $v \in V^\Gamma$ and $I \subseteq S$ a spherical Γ-orbit, we have*

$$w_0(I)v = v - \alpha_I^\vee(v)\alpha_I \in V^\Gamma,$$

where $\alpha_I^\vee : V^\Gamma \to \mathbb{R}$ is a linear map satisfying $\alpha_I^\vee(\alpha_I) = 2$.

Proof. For $v \in V^\Gamma$, we have $v - w_0(I)v \in V_I \cap V^\Gamma$. The latter intersection is just $\mathbb{R}\alpha_I$ by choice of I. So we get a linear form $\alpha_I^\vee : V^\Gamma \to \mathbb{R}$ such that $w_0(I)v = v - \alpha_I^\vee(v)\alpha_I$ for all $v \in V^\Gamma$.

Since $\alpha_I, w_0(I)\alpha_I \in V^\Gamma \cap V_I$ and $w_0(I)^2 = 1$, we see that $w_0(I)\alpha_I = \pm\alpha_I$. It cannot be $+\alpha_I$, hence $\alpha_I^\vee(\alpha_I) = 2$. □

Lemma 97. *Let $I_1, \ldots, I_n$ be spherical Γ-orbits, $w = w_0(I_1)\cdots w_0(I_n)$ and $s \in S$ such that $w\alpha_s \leqslant 0$. Denote by I the Γ-orbit containing s. Then I is spherical and*

$$w = w_0(I_1)\cdots w_0(I_{i-1})w_0(I_{i+1})\cdots w_0(I_n)w_0(I)$$

for some index $i \in \{1, \ldots, n\}$.

Proof. We saw that I is spherical in Lemma 94. We may certainly find an index i such that

$$w_0(I_{i+1})\cdots w_0(I_n)\alpha_s \geqslant 0, \qquad w_0(I_i)\cdots w_0(I_n)\alpha_s \leqslant 0.$$

Since $w' = w_0(I_{i+1})\cdots w_0(I_n)$ is in W^Γ, we see $w'\alpha_{s'} \in \operatorname{inv}(w_0(I_i))$ for all $s' \in I$.

Conversely, if $(w')^{-1}\alpha_{s'} \leqslant 0$ for some $s' \in I_i$, then $(w')^{-1}\beta \leqslant 0$ for all $\beta \in \operatorname{inv}(w_0(I_i))$ (using $(w')^{-1} \in W^\Gamma$). This contradicts $w'\alpha_s \in \operatorname{inv}(w_0(I_i))$. We see $(w')^{-1}\alpha_{s'} \geqslant 0$ for all $s' \in I_i$. The same argument shows $(w'w_0(I))^{-1}\alpha_{s'} \leqslant 0$ for all $s' \in I_i$. Hence $(w')^{-1}(\alpha_{s'}) \in \operatorname{inv}(w_0(I))$.

Looking at minimal elements in the respective inversion sets, we see that w' sends $\{\alpha_{s'} \mid s' \in I\}$ bijectively to $\{\alpha_{s'} \mid s' \in I_i\}$. Hence $I_i = w'I(w')^{-1}$ and the claim follows. □

Theorem 98. *(a) (W^Γ, S^Γ) is a Coxeter group.*

(b) The action of W^Γ on V^Γ is contains the geometric representation of (W^Γ, S^Γ) as natural subrepresentation.

(c) For spherical Γ-orbits $I_1, \ldots, I_n$, the product

$$w_0(I_1)\cdots w_0(I_n)$$

is length-additive in W if and only if it is reduced in W^Γ.

(d) W^Γ inherits the Bruhat order, left weak order and right weak order from W.

Proof. For clarity, we denote the length function of W by $\ell_W : W \to \mathbb{Z}$.

(a) We saw that W^Γ is generated by S^Γ and by construction, each element in S^Γ is an involution. Denote by $\ell_{W^\Gamma} : W^\Gamma \to \mathbb{Z}$ the corresponding length function.

It remains to verify the Weak Exchange Condition.

Let $w \in W^\Gamma$ be written as $w = w_0(I_1) \cdots w_0(I_n)$ for spherical Γ-orbits $I_1, \ldots, I_n$ with n minimal. Let I be another spherical Γ-orbit such that $\ell_{W^\Gamma}(ww_0(I)) \leqslant \ell_{W^\Gamma}(w)$. If $w\alpha_s \leqslant 0$ for some $s \in I$, we are done by the previous lemma. Otherwise, we find $ww_0(I)\alpha_s \leqslant 0$, and then the previous lemma shows

$$\ell_{W^\Gamma}(ww_0(I)) > \ell_{W^\Gamma}(w),$$

contradiction.

(b) Let $\tilde{V}$ be the vector space with a formal basis $\beta_{w_0(I)}$ where I runs through the spherical Γ-orbits. Choose real numbers $k_{w_0(I),w_0(I')} := \alpha_I^\vee(\alpha_{I'})$. From the fact that $W^\Gamma \to \mathrm{GL}(V^\Gamma)$ is a well-defined and faithful representation of W^Γ, one deduces that the numbers $k_{w_0(I),w_0(I')}$ as defined above satisfy the properties needed to define a geometric representation of W^Γ.

Then the natural map $\tilde{V} \hookrightarrow V^\Gamma, \beta_{w_0(I)} \mapsto \alpha_I$ is a W^Γ-equivariant injection, by construction.

(c) If it is not length additive, it cannot be reduced for (W^Γ, S^Γ) using Lemma 28 for (W, S) together with the previous Lemma. Conversely, if it is not reduced for (W^Γ, S^Γ), we find some index with $w_0(I_1) \cdots w_0(I_{i-1})\alpha_{I_i} \leqslant 0$ using Theorem 26 for (W^Γ, S^Γ), and then the product cannot be length additive by Lemma 28.

(d) For $v, w \in W^\Gamma$, we have

$$\ell_W(wv) = \ell_W(w) + \ell_W(v) \iff \ell_{W^\Gamma}(wv) = \ell_{W^\Gamma}(w) + \ell_{W^\Gamma}(v)$$

by part (c). Hence the statements on the weak orders follow. If $v \leqslant w$ with respect to the Bruhat order on W^Γ, we conclude that the same holds in the Bruhat order on W using part (c) and the word criterion for the Bruhat order.

For the converse implication, we use induction on w and compare the statements of Corollary 61 for (W, S) with (W^Γ, S^Γ). ☐

We call (W^Γ, S^Γ) the *folding* of (W, S) under the action of Γ. The Coxeter diagram of (W^Γ, S^Γ) can be determined using the following result:

Proposition 99. *Let I, J be spherical Γ-orbits and consider the parabolic subgroup $W_{I \cup J}$.*

(a) $W_{I\cup J}$ is finite if and only if the order $m := m(w_0(I), w_0(J))$ of $w_0(I)w_0(J)$ in W^Γ is finite.

(b) If m is finite, then

$$m = \frac{2\ell(w_0(I \cup J))}{\ell(w_0(I)) + \ell(w_0(J))}.$$

(c) If m is finite and odd, then $\ell(w_0(I)) = \ell(w_0(J))$.

Proof. If $w \in W_{I \cup J}$ is a longest element, then $w \in W^\Gamma$ since there can be only one longest element. Moreover, we have $w\alpha_I, w\alpha_J \leqslant 0$. Thus $w \in W^\Gamma$ is a longest element as well.

If conversely $w \in (W_{I \cup J})^\Gamma$ is a longest element, we must have $w\alpha_s \leqslant 0$ for all $s \in I$ and all $s \in J$. Thus $w \in W_{I \cup J}$ is a longest element as well.

Now $(W^\Gamma_{I \cup J}, \{w_0(I), w_0(J)\})$ is a dihedral group. It is finite if and only if the order of $w_0(I)w_0(J)$ is finite. In this case, the longest element is given by the alternating product of m terms

$$w_0 = w_0(I)w_0(J)w_0(I) \cdots .$$

This expression must be length additive for (W, S) by the theorem. We conclude

$$\ell(w_0) = \lceil m/2 \rceil \ell(w_0(I)) + \lfloor m/2 \rfloor \ell(w_0(J)).$$

We may repeat the argument with I and J interchanged. So if m is odd, we must have $\ell(w_0(I)) = \ell(w_0(J))$ and in any case we obtain

$$\ell(w_0) = \frac{m}{2} \left(\ell(w_0(I)) + \ell(w_0(J)\right).$$

This finishes the proof. □

Exercise 100. Consider our standard example $W = S_3$ and let $\Gamma = \{1, w_0\}$, acting on (W, S) by conjugation. Determine the Coxeter diagram for (W^Γ, S^Γ).

15 List of finite Coxeter groups

It is a fundamental result in the theory of Coxeter groups that we can enumerate all finite examples.

Definition 101. A Coxeter group (W, S) is called *reducible* if there exist non-empty subsets $S_1, S_2 \subseteq S$ with $S = S_1 \sqcup S_2$ such that $s_1 s_2 = s_2 s_1$ for all $s_1 \in S_1, s_2 \in S_2$. Otherwise, it is called *irreducible*.

If (W, S) is reducible as above, then naturally $W = W_{S_1} \times W_{S_2}$ and W is finite iff both W_{S_1} and W_{S_2} are. In this section, we describe all finite and irreducible Coxeter groups.

Example 102. For $n \geqslant 1$, the Coxeter group associated with the diagram of type

$$A_n : \bullet - \bullet - \cdots - \bullet$$

is S_{n+1}. The length of w_0 is given by $\frac{n(n+1)}{2}$.

For $n \geqslant 2$, the diagram A_n has a unique non-trivial automorphism. With Γ generated by this automorphism, S^Γ consists of $\lceil n/2 \rceil$ elements. The quotient is of type $B_{\lceil n/2 \rceil}$.

Example 103. For $n \geqslant 2$, consider the Coxeter diagram

$$B_n : \bullet \overset{4}{-} \bullet - \bullet - \cdots - \bullet.$$

The Coxeter group of signed permutations was already introduced. The length of the longest element is n^2.

The diagram has no non-trivial automorphisms unless $n = 2$, in which case there is one non-trivial automorphism (whose folding yields A_1).

Example 104. For $n \geqslant 4$, consider the Coxeter diagram

$$D_n : \quad \begin{tikzcd} \bullet \arrow[dr, dash] & & & & \\ & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \\ \bullet \arrow[ur, dash] & & & & \end{tikzcd} .$$

If (W, S) is of type B_n with s_0 being the isolated node on the left (i.e. $m(s_0, s) \in \{2, 4\}$ for $s \in S$), then we get a Coxeter group of type D_n by considering the subgroup of W consisting of those elements $w \in W$ where the number of occurrences of s_0 in some / any reduced word of w is even.

The longest element of type D_n has length $n^2 - n$.

There is a non-trivial automorphism interchanging the two leftmost nodes in the above picture. The corresponding folding is of type B_{n-1}. If $n \geqslant 5$, this is the only non-trivial automorphism.

For $n = 4$, the automorphism group of the Coxeter diagram is the symmetric group S_3, acting freely on the three outer nodes. If $1 \neq \Gamma \leqslant S_3$ fixes one of the outer nodes, the folding is B_3. Otherwise, there are only two Γ-orbits and the resulting Coxeter group has type $G_2 = I_2(6)$.

Example 105. For $n = 6, 7, 8$, consider the Coxeter diagram

$$E_n : \quad \begin{tikzcd} & & \bullet \arrow[d, dash] & & & \\ \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \end{tikzcd} .$$

The corresponding Coxeter groups does not have any easily accessible description, but it is finite with longest element of length $36, 63, 120$ respectively.

The automorphism group is trivial unless $n = 6$, in which case there is precisely one non-trivial automorphism (in the above picture, its the left-right symmetry). The corresponding quotient has type F_4.

Example 106. Consider the Coxeter diagram

$$F_4 : \bullet - \bullet \overset{4}{-} \bullet - \bullet .$$

The corresponding Coxeter group is finite with longest element having length 24. There is a unique automorphism, and the resulting folding is of type $I_2(8)$.

Example 107. The Coxeter diagrams

$$H_3: \bullet\overset{5}{-}\bullet-\bullet, \quad H_4: \bullet\overset{5}{-}\bullet-\bullet-\bullet$$

yield finite Coxeter groups with longest elements of length 15 resp. 60. There are no non-trivial diagram automorphisms.

Example 108. Let $m \geqslant 3$ and consider the Coxeter diagram

$$I_2(m): \bullet\overset{m}{-}\bullet\,.$$

The corresponding Coxeter group is the dihedral group of order $2m$, with longest element having length m. There is a unique non-trivial diagram automorphism, whose folding has type A_1.

Remark 109. Call a Coxeter group *crystallographic*[2] if it is possible to choose the constants $k_{s,s'}$ defining the geometric representation to be integers. This is equivalent to $m(s,s') \in \{1,2,3,4,6,+\infty\}$ for all $s,s' \in S$.

The finite crystallographic Coxeter groups are precisely the *Weyl groups* known from Lie theory. The crystallographic dihedral groups are $I_2(3) = A_2, I_2(4) = B_2$ and $I_2(6)$ which is called G_2 for this reason.

If (W,S) is such a Weyl group, there is typically only one way to choose the constants $k_{s,s'} \in \mathbb{Z}$ up to a diagram automorphism. The only exception is the case B_n, in which case there are essentially two ways. These choices yield root systems called B_n and C_n, whose Weyl groups are isomorphic as Coxeter groups.

In the exercises of this section, you are allowed to use without proof that the above list of finite and irreducible Coxeter groups is complete.

Exercise 110. Let (W,S) be a Coxeter group of type B_n. Using the description in terms of signed permutations, give a description of the Bruhat order similar to Theorem 76.

Exercise 111. Call a Coxeter group *simply laced* if $m(s,s') \in \{1,2,3\}$ for all $s,s' \in S$.

(a) Show that a finite Coxeter group is a Weyl group if and only if it is the folding of a simply laced finite Coxeter group.

(b) Look up a classification of affine Coxeter groups, e.g. [BB05, Appendix A1] or wikipedia[3] and show that each affine Coxeter group is the folding of an affine and simply laced Coxeter group.

(c) Show that the folding of an irreducible affine Coxeter group is again affine or finite.

Exercise 112. Suppose that (W,S) is an irreducible Coxeter group and Γ a group of automorphisms such that (W^Γ, S^Γ) is of type H_3, H_4 or $I_2(m)$ for $m \neq 3,4,6$ (i.e. one of the finite non-Weyl groups). Show that $W = W^\Gamma$.

[2] There are other definitions of this notion, e.g. [Hum90, Section 6.6].

[3] `https://en.wikipedia.org/wiki/Affine_root_system`.

16 Iwahori-Hecke algebra

Remember that for a Lie group G with Iwahori B and Weyl group W, we have

$$G = \bigsqcup_{w \in W} BwB.$$

The *Hecke algebra* of G is defined as

$$\mathcal{H}(G) = \{f : G \to \mathbb{C} \mid f(b_1 g b_2) = f(g) \quad \forall g \in G, b_1, b_2 \in B\}.$$

It becomes a $\mathbb{C}$-algebra where we define multiplication via a convolution product

$$(f_1 f_2)(g) = \int_G f_1(g(g')^{-1}) f_2(g') \, dg'.$$

As vector space over $\mathbb{C}$, it has a basis given by the indicator functions $T_w = \mathbb{1}_{BwB}$ for $w \in W$. Assume that all subsets BsB for $s \in S$ have the same volume $q \in \mathbb{C}$. Then one computes

$$T_w T_s = \begin{cases} T_{ws}, & ws > w, \\ qT_{ws} + (q-1)T_w, & ws < w. \end{cases}$$

It is common to change variables $\tilde{T}_w = q^{-\ell(w)/2} T_w$, which then satisfy

$$\tilde{T}_w \tilde{T}_s = \begin{cases} \tilde{T}_{ws}, & ws > w, \\ \tilde{T}_{ws} + (q^{1/2} - q^{-1/2})\tilde{T}_w, & ws < w. \end{cases}$$

Personally, I like to introduce the shorthand notation $Q = q^{1/2} - q^{-1/2}$. In any case, there are many different but largely equivalent parametrizations of the Hecke algebra around. Confer to [Bon17] for a most general treatment.

This is a bit of a toy example, really one wants to use p-adic Lie groups and compact subgroups instead of (G, B). Anyway, the goal of this section is to introduce an analogue construction for arbitrary Coxeter groups.

Definition 113. Let A be an commutative ring and $q_1, q_2 \in A$. The *Hecke algebra with equal parameters* $\mathcal{H} = \mathcal{H}(W)$ associated to (W, A, q_1, q_2) is the A-algebra generated by

$$T_w, \qquad w \in W$$

and relations

$$\begin{aligned} T_w T_{w'} =& T_{ww'}, && \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ T_s^2 =& q_1 T_1 + q_2 T_s, && \text{if } s \in S. \end{aligned}$$

Lemma 114. *(a) If $q_1 \in A$ is invertible, then each $T_w \in \mathcal{H}(W)$ is invertible in $\mathcal{H}(W)$. If $w = s$ is a simple reflection, its inverse is given by*

$$T_s^{-1} = q_1^{-1}(T_s - q_2).$$

(b) *For any $w \in W$ and $s \in S$, we have*

$$T_w T_s = \begin{cases} T_{ws}, & ws > w \\ q_1 T_{ws} + q_2 T_w, & sw < w. \end{cases} . T_s T_w = \begin{cases} T_{sw}, & sw > w \\ q_1 T_{sw} + q_2 T_w, & sw < w. \end{cases}$$

Proof. (a) The formula for T_s^{-1} is immediately verified. In general, T_w is a product of T_{s_i} following a fixed reduced word of w.

(b) The claims are clear if $ws > w$ resp. $sw > w$. If $ws < w$, we get a length additive product $w = (ws) \cdot s$, hence

$$T_w T_s = T_{ws} T_s^2 = T_{ws}(q_1 T_1 + q_2 T_s) = q_1 T_{ws} + q_2 T_w.$$

The other part is analogous. ☐

The following structural result is crucial for the remainder of the course.

Proposition 115. *As module over A, the Hecke algebra $\mathcal{H}(W)$ is free over the basis given by T_w for $w \in W$.*

For the proof of the proposition, we define the A-module M to be free over W,

$$M := \bigoplus_{w \in W} A v_w.$$

For each $s \in S$, denote by $\tau_s \in \mathrm{GL}(M)$ the linear automorphism defined on basis vectors by

$$\tau_s(v_w) = \begin{cases} v_{sw}, & sw > w \\ q_1 v_{sw} + q_2 v_w, & sw < w. \end{cases}$$

Lemma 116. *Let $s, t \in S$ such that $m(s,t) < +\infty$. Let τ, τ' be the $m(s,t)$-fold compositions*

$$\tau = \cdots \circ \tau_s \circ \tau_t \circ \tau_s$$
$$\tau' = \cdots \circ \tau_t \circ \tau_s \circ \tau_t.$$

Then $\tau = \tau'$ as endomorphisms of M.

Proof. We show $\tau(v_w) = \tau'(v_w)$ by induction on $\ell(w)$. Write $w_0(s,t)$ for the longest element of W_J where $J = \{s,t\}$.

If $sw > w$ and $tw > w$, we get $\ell(w_0(s,t)w) = \ell(w_0(s,t)) + \ell(w)$. Hence we directly evaluate $\tau(v_w) = v_{w_0(s,t)w} = \tau'(v_w)$.

Otherwise, we may and do assume without loss of generality that $sw < w$. Let $s', t' \in \{s,t\}$ such that the two reduced decompositions of $w_0(s,t)$ have the form

$$s't's' \cdots sts = w_0(s,t) = t's't' \cdots tst.$$

Then we evaluate

$$\begin{aligned}\tau(v_w) =& \tau_{s'} \circ \tau_{t'} \circ \cdots \circ \tau_t \circ \tau_s(v_w) = \tau_{s'} \circ \tau_{t'} \circ \cdots \circ \tau_t(q_1 v_{sw} + q_2 v_w) \\ =& q_1 \tau_{s'} \circ \tau_{t'} \circ \cdots \circ \tau_t(v_{sw}) + q_2 \tau_{s'} \circ \tau_{t'} \circ \cdots \circ \tau_t \circ \tau_s(v_{sw}) \\ \underset{\text{ind}}{=}& q_1 \tau_{s'} \circ \tau_{t'} \circ \cdots \circ \tau_t(v_{sw}) + q_2 \tau_{t'} \circ \tau_{s'} \circ \cdots \circ \tau_s \circ \tau_t(v_{sw}).\end{aligned}$$

It is a simple calculation to verify that $\tau_{t'}^2(v_z) = q_1 v_z + q_2 \tau_{t'}(v_z)$ holds for all $z \in W$. Thus we calculate similarly

$$\begin{aligned}\tau'(v_w) =& \tau' \tau_s(v_{sw}) = \tau_{t'} \circ \underline{\tau_{s'} \cdots \circ \tau_s \circ \tau_t \circ \tau_s(v_{sw})} \\ \underset{\text{ind.}}{=}& \tau_{t'} \circ \underline{\tau_{t'} \circ \tau_{s'} \cdots \circ \tau_s \circ \tau_t(v_{sw})} \\ =& q_1 \tau_{s'} \cdots \circ \tau_s \circ \tau_t(v_{sw}) + q_2 \tau_{t'} \circ \tau_{s'} \cdots \circ \tau_s \circ \tau_t(v_{sw}).\end{aligned}$$

This finishes the induction and the proof. □

By Matsumoto's theorem, we get that whenever $w = s_1 \cdots s_n$ is a reduced word, the map

$$\tau_w = \tau_{s_1} \circ \cdots \circ \tau_{s_n}$$

depends only on w and not the particular chosen word.

Proof of Proposition 115. We get a map from $\mathcal{H}(W)$ to the endomorphism algebra $\mathrm{End}_A(M)$ sending T_w to τ_w. We get a map from $\mathrm{End}_A(M)$ to M sending f to $f(v_1)$. Composing both maps, we get an A-linear map from $\mathcal{H}(W)$ to M sending T_w to v_w. Hence the T_w are linearly independent. They certainly generate $\mathcal{H}(W)$ as A-module. □

Exercise 117. Compute the product $T_{w_0} \cdot T_{w_0}$ for the Hecke algebras of S_3 and S_4, with A being the polynomial ring $A = \mathbb{Z}[q_1, q_2]$.

Exercise 118. Show that for all $w, w' \in W$, the value $T_w T_{w'} - T_{ww'}$ is divisible by q_2 in $\mathcal{H}(W)$.

Exercise 119. Let $w_1, w_2 \in W$ and express the product $T_{w_1} T_{w_2}$ using the usual basis as

$$T_{w_1} T_{w_2} = \sum_{w_3 \in W} f_{w_3} T_{w_3}, \qquad f_\bullet \in A = \mathbb{Z}[q_1, q_2].$$

Show that if $\ell(w_1) + \ell(w_2) + \ell(w_3)$ is even (resp. odd), then only even (resp. odd) powers of q_2 occur in the polynomial $f_{w_3} \in \mathbb{Z}[Q]$.

17 Braid group

Definition 120. Associated to a Coxeter system (S, m), we associate the *Braid group* B presented by the generators S and relations

$$\underbrace{ss'ss'\cdots}_{m(s,s')\text{ terms}} = \underbrace{s'ss's\cdots}_{m(s,s')\text{ terms}}.$$

If (S, m) is of type A_{n-1}, we denote B by B_n.

Certainly the Coxeter group W is a quotient of B, so S_n is a quotient of B_n. The braid group B_n is infinite in general. It has the following geometric interpretation:

Consider a sequence of smooth paths in $[0, n] \times [0, 1]$

$$\begin{aligned} p_0 :& (1,0) \to (0,0), \\ p_1 :& (1,1) \to (0,1), \\ & \vdots \\ p_n :& (1,n) \to (0,n) \end{aligned}$$

such that any two paths p_i, p_j intersect only at a finite number of points. To each such crossing, we moreover record information on which path lies *above* the other one. (picture)

Such a diagram, up to smooth stretching and moving of paths above and below each other, can be called a *braid*. These form a group by composition (picture). We can associate $s_i \in S$ to the braid consisting of straight paths $p_j : (1, j) \to (0, j)$ for $j \neq i, i+1$, $p_i : (1, i) \to (0, i+1)$ and $p_{i+1} : (1, i+1) \to (0, i)$ such that p_i lies above p_{i+1} (picture). The resulting group of braids is isomorphic to B_n.

Each braid can be associated to a union of knots, or links, in $\mathbb{R}^3$ by connecting $(0, i)$ to $(1, i)$ outside of the box $[0, n] \times [0, 1]$ (picture). The same link may have different braid presentations, but there is some sort of equivalence relations between braids.

So the knot theorists are interested in representations of the group B_n, say over the complex so $B_n \to \mathrm{GL}_n(\mathbb{C})$. Any two generators of B_n are conjugate, so representations with only one eigenvalue over $\mathbb{C}$ (e.g. one-dimensional representations) must be of the form $s_i \mapsto \mathrm{diag}(c, \dots, c)$ for some constant $c \in \mathbb{C}^\times$.

Let us consider a representation $\rho : B_n \to \mathrm{GL}_n(\mathbb{C})$ such that each $\rho(s_i)$ is diagonalizable with at most two eigenvalues, e.g. for $n = 2$. By conjugation, they have the same eigenvalues, say λ_1, λ_2. We get

$$\begin{aligned} & (\rho(s_i) - \lambda_1)(\rho(s_i) - \lambda_2) = 0, \\ \Longleftrightarrow\ & \rho(s_i)^2 = (\lambda_1 + \lambda_2)\rho(s_i) - \lambda_1\lambda_2. \end{aligned}$$

The fact that ρ is a representation is equivalent to

$$\begin{aligned} \rho(s_i)\rho(s_j) =& \rho(s_j)\rho(s_i) \text{ if } |i - j| \geqslant 2 \\ \rho(s_i)\rho(s_{i+1})\rho(s_i) =& \rho(s_{i+1})\rho(s_i)\rho(s_{i+1}). \end{aligned}$$

If we put $q_1 = -\lambda_1\lambda_2$ and $q_2 = \lambda_1 + \lambda_2$, these representations are the same as representations of the Hecke algebra $\mathcal{H}(S_n)$ defined over $\mathbb{C}$. We refer to the very remarkable article [Jon87] using this correspondence to define an important invariant of links.

18 Kazhdan-Lusztig polynomials

We specialize to the Hecke algebra defined over $A = \mathbb{Z}[q^{\pm 1/2}]$ with parameters $q_1 = q$ and $q_2 = q - 1$. In other words, $(T_s + 1)(T_s - q) = 0$ for all $s \in S$.

Kazhdan-Lusztig construct a large class of irreducible representations of $\mathcal{H}(W)$, which we review in this section. In particular, specialized to $W = S_n$ and $q = 1$, all irreducible representations of the group algebra $\mathbb{C}[S_n]$ arise from their construction.

The ring A has an automorphism induced from $q^{1/2} \mapsto q^{-1/2}$ which we denote by $a \mapsto \overline{a}$.

Lemma 121. *There is an automorphism of rings $\mathcal{H}(W) \to \mathcal{H}(W)$ sending*

$$aT_w \mapsto \overline{aT_w} := \overline{a}T_{w^{-1}}^{-1}.$$

Proof. It suffices to see that the $\overline{T_w}$ satisfy the defining relations of the Iwahori-Hecke algebra. If $\ell(ww') = \ell(w) + \ell(w')$, we get

$$\overline{T_w} \cdot \overline{T_{w'}} = T_{w^{-1}}^{-1}T_{(w')^{-1}}^{-1} = (T_{(w')^{-1}}T_{w^{-1}})^{-1} = T_{(ww')^{-1}}^{-1} = \overline{T_{ww'}}.$$

Let now $s \in S$. We compute

$$\overline{T_s} \cdot \overline{T_s} = T_s^{-1}T_s^{-1}.$$

Observe $T_s^{-1} = q^{-1}(T_s + 1 - q)$ so that

$$\begin{aligned}(T_s^{-1})^2 =& q^{-2}(T_s^2 + 2(1-q)T_s + (1-q)^2) = q^{-2}((q-1)T_s + q + 2(1-q)T_s + (1-q)^2)\\ =& q^{-2}((1-q)(T_s + 1 - q) + q) = q^{-1}(1-q)T_s^{-1} + q^{-1} = \overline{q-1} \cdot \overline{T_s} + \overline{q}.\end{aligned}$$

So the relations are satisfied for $\overline{T_s} \cdot \overline{T_s}$. This finishes the proof. □

Definition 122. For $w \in W$, we write

$$\overline{T_w} = \sum_{v \in V} \overline{R_{v,w}} q^{-\ell(v)} T_v \in \mathcal{H}(W)$$

for polynomials $R_{v,w} \in A$, called the *R-polynomial* associated with $v, w \in W$.

They satisfy the following properties.

Lemma 123. *Let $v, w \in W$ and $s \in S$.*

(a) If $ws < w$ then

$$R_{v,w} = \begin{cases} R_{vs,ws}, & \text{if } vs < v,\\ qR_{vs,ws} + (q-1)R_{v,ws}, & \text{if } vs > v.\end{cases}$$

(b) If $w = 1$, then

$$R_{v,1} = \begin{cases} 1, & \text{if } v = 1,\\ 0, & \text{if } v \neq 1.\end{cases}$$

(c) If $v \leqslant w$ in the Bruhat order, then $R_{v,w}$ is a monic polynomial of degree $\ell(w) - \ell(v)$ in $\mathbb{Z}[q]$. Otherwise, $R_{v,w} = 0$.

Proof. (a) We compute

$$\sum_{u\in W} \overline{R_{u,w}} q^{-\ell(u)} T_u = \overline{T_w} = \overline{T_{ws}} \cdot \overline{T_s} = \sum_{u\in W} \overline{R_{u,ws}} q^{-\ell(u)} T_u T_s^{-1}$$
$$= \sum_{\substack{u\in W\\ us<u}} \overline{R_{u,ws}} q^{-\ell(u)} T_{us} + \sum_{\substack{u\in W\\ us>u}} \overline{R_{u,ws}} q^{-\ell(u)} (\overline{q-1} T_u + \overline{q} T_{us}).$$

Let us compare the A-coefficients for T_v in the above expression. If $vs < v$, then the left sum does not contribute to T_v and the right sum contributes the value $\overline{R_{vs,ws}} q^{-\ell(v)} T_v$ coming from $u = vs$.

If $vs > v$, then the left sum contributes $\overline{qR_{vs,ws}} q^{-\ell(v)} T_v$ coming from $u = vs$ and the right sum contributes $\overline{(q-1)R_{v,ws}} q^{-\ell(v)} T_v$ coming from $u = v$.

(b) Indeed we have $\overline{T_1} = T_1$.

(c) Induction on $\ell(w)$, the inductive start for $w = 1$ being clear. In the inductive step, pick $s \in S$ such that $ws < w$ and let $v' = \min(v, vs)$. By the lifting property of Bruhat order, $v \leqslant w$ if and only if $v' \leqslant ws$.

So in case $v \nleqslant w$, we get $v' \nleqslant ws$ and also $v \nleqslant ws$, showing that $R_{v,w}$, being a linear combination of $R_{v',ws}$ and $R_{v,ws}$, is zero.

If $v \leqslant w$, then by induction we see that $R_{v',ws}$ is a monic polynomial of degree $\ell(ws) - \ell(v')$. If $vs < v$, we get $R_{v,w} = R_{v',ws}$ and $\ell(sw) - \ell(v') = \ell(w) - \ell(v)$, proving the claim. Otherwise, we get

$$R_{v,w} = qR_{vs,ws} + (q-1)R_{v',ws}$$

with $R_{vs,ws}$ having degree $\leqslant \ell(ws) - \ell(vs) = \ell(w) - \ell(v) - 2$ and $R_{v',ws}$ being monic of degree $\ell(ws) - \ell(v') = \ell(w) - \ell(v) - 1$. The induction is complete. □

Theorem 124. *There is a unique basis $\{C_w\}_{w\in W}$ of the A-module $\mathcal{H}(W)$ satisfying the following properties: $\overline{C_w} = C_w$ for all $w \in W$ and*

$$C_w = \sum_{v\leqslant w} (-1)^{\ell(v)+\ell(w)} q^{\ell(w)/2-\ell(v)} \overline{P_{v,w}} T_v \in \mathcal{H}(W)$$

for polynomials $P_{v,w} \in A$ of q-degree $\leqslant 1/2(\ell(w) - \ell(v) - 1)$ such that $P_{w,w} = 1$.

Proof sketch. Uniqueness: Using the definition of R-polynomials, the condition $\overline{C_w} = C_w$ expanded as required implies for $u < w$ that

$$q^{(\ell(w)-\ell(u))/2} \overline{P_{u,w}} - q^{(\ell(u)-\ell(w))/2} P_{u,w} = \sum_{u<v\leqslant w} (-1)^{\ell(u)+\ell(v)} q^{\ell(v)-\ell(u)/2-\ell(w)/2} \overline{R_{u,v}} P_{v,w}.$$

In an inductive sense, we may assume that the $P_{v,w}$ are uniquely determined for $u < v \leqslant w$. Now

$$q^{(\ell(u)-\ell(w))/2} P_{u,w}$$

is a polynomial in $q^{-1/2}$ without constant term. Applying the involution $\bar{\cdot}$ to it yields a polynomial in $q^{1/2}$ without constant term. There are no cancellations between such polynomials possible, so $P_{u,w}$ is uniquely determined by the $P_{v,w}$ for $u < v \leqslant w$.

Existence: Use the above relation to construct suitable elements C_w which are invariant under the involution $\bar{\cdot}$ and have the required shape. Then the basis property is easily verified. □

Definition 125. The polynomials $P_{v,w} \in \mathbb{Z}[q]$ are called *Kazhdan-Lusztig polynomials.*

Exercise 126. Calculate the R-polynomials for $W = S_3$. With the given relation in the above proof, calculate the Kazhdan-Lusztig polynomials.

Exercise 127. Verify the full proof of existence and uniqueness of Kazhdan-Lusztig polynomials [KL79, Section 2.2].

Exercise 128. Show that $P_{v,w} \neq 0$ whenever $v \leqslant w$.

19 Kazhdan-Lusztig representations

We continue with the notation from the previous section.

Definition 129. For $v \leqslant w$ in W, let $\mu(v, w)$ be the coefficient of $q^{(\ell(w)-\ell(v)-1)/2}$ in $P_{v,w}$. If $v > w$, we define $\mu(v, w) := \mu(w, v)$.

We write $v \prec w$ if $v < w$, $\ell(w) - \ell(v)$ is odd and $\mu(v, w) \neq 0$.

Lemma 130. *Let $s \in S$ and $w \in W$.*

(a) If $sw < w$, then $T_sC_w = -C_w$.

(b) If $sw > w$, then

$$q^{-1/2}T_sC_w = C_{sw} + q^{1/2}C_w + \sum \mu(z, w)C_z,$$

with the sum taken over all $z \prec w$ with $sz < z$.

Proof reference. Cf. [KL79, Sections 2.2, 2.3]. □

We see that the basis elements $C_w \in \mathcal{H}(W)$ are not invertible, unlike their T_w counterparts. This is the exciting aspect from a representation theoretic point of view!

Definition 131. We define the *left cell order* $\leqslant^L$ on W to be the partial order generated by the relations $v <^L w$ whenever the following two conditions are satisfied:

(a) $v \prec w$ or $w \prec v$ and

(b) there exists $s \in S$ with $sv < v$ and $sw > w$.

The *right cell order* is defined by $v \leqslant^R w$ if $v^{-1} \leqslant^L w$. It has an analogue description if we replace (b) by $vs < v$ and $ws > w$.

The *two-sided cell order* is the partial order generated by all pairs $v \leqslant^{LR} w$ such that $v \leqslant^L w$ or $v \leqslant^R w$.

For $\bullet \in \{L, R, LR\}$, we write $v \sim^\bullet w$ if $v \leqslant^\bullet w \leqslant^\bullet v$.

It is rather straightforward from Lemma 130 to see the following:

Proposition 132. *Let $w \in W$.*

(a) The left ideal $\mathcal{H}(W)C_w \subseteq \mathcal{H}(W)$ is a free A-module with basis given by all C_v for $v \leqslant^L w$.

(b) The right ideal $C_w\mathcal{H}(W) \subseteq \mathcal{H}(W)$ is a free A-module with basis given by all C_v for $v \leqslant^R w$.

(c) The two-sided ideal $\mathcal{H}(W)C_w\mathcal{H}(W) \subseteq \mathcal{H}(W)$ is a free A-module with basis given by all C_v for $v \leqslant^{LR} w$. □

Definition 133. The equivalence classes of $\sim^L, \sim^R, \sim^{LR}$ are called *left/right/two sided cells* in W. Given such a left cell C, the *left cell module* for C is given by

$$(\mathcal{H}(W)w)/I$$

for some $w \in W$ where I is the left ideal in $\mathcal{H}(W)$ generated by all C_v for $v <^L w$. Similarly, right cells induce right cell modules and two-sided cells induce cell bimodules.

We may consider the case of a finite Coxeter group and specialize $A \to \mathbb{C}, q \mapsto 1$.

For the $W = S_n$ being the symmetric group, it follows that the left cell representations are irreducible and cover all irreducible complex representations. For general finite Coxeter groups, these complex representations may be reducible, but contain every irreducible representation as a subrepresentation.

The theory of Kazhdan-Lusztig polynomials, cells and representations is subject to active research and many conjectures.

20 Reflection orders

The concept of reflection order appears in different places of the theory of Coxeter groups. These orders can be used e.g. to give a convenient description of R-polynomials.

Definition 134. A *reflection order* $\prec$ on Φ^+ is a total order such that for all $\alpha, \beta \in \Phi^+$ with $\alpha \prec \beta$ and scalars $\lambda, \mu \in \mathbb{R}_{>0}$ satisfying $\lambda\alpha + \mu\beta \in \Phi^+$, we have

$$\alpha \prec \lambda\alpha + \mu\beta \prec \beta.$$

Example 135. For $W = S_n$, the positive roots can be identified as

$$\Phi^+ = \{e_i - e_j \mid i < j\} \subseteq \mathbb{R}^n.$$

Then the *lexicographic order* is a reflection order on Φ^+, i.e.

$$(e_i - e_j) \prec (e_{i'} - e_{j'}) \iff (i < i') \text{ or } (i = i' \text{ and } j < j').$$

Indeed, if $\alpha \prec \beta$ are such that $\alpha + \beta \in \Phi^+$, we must have $\alpha = e_i - e_j, \beta = e_j - e_k$ for indices $i < j < k$. Now $\alpha + \beta = e_i - e_k$ sits between α and β as claimed. There are many more reflection orders on Φ^+.

The first non-trivial observation of the theory of reflection orders is the following:

Proposition 136. *There exists a reflection order $\prec$ for every Coxeter group (W,S).*

Proof. Choose a well-ordering $<$ on S and for each $s \in S$ a positive scalar $\lambda_s > 0$. For $\alpha = \sum_{s\in S} c_s\alpha_s \in \Phi^+$ we define

$$v(\alpha) = (\lambda_s c_s)_{s\in S} \in \mathbb{R}^S_{\geqslant 0},\ \ \sigma(\alpha) = \sum(v(\alpha)) \in \mathbb{R}_{>0} \text{ and } \tilde{v}(\alpha) = \frac{1}{\sigma(\alpha)}v(\alpha) \in \mathbb{R}^S_{\geqslant 0}.$$

We define the order $\prec$ on Φ^+ by comparing the vectors $\tilde{v}(\alpha)$ lexicographically. Explicitly, $\alpha \prec \beta$ if and only if there exists $s \in S$ such that $\tilde{v}(\alpha)_s < \tilde{v}(\beta)_s$ and

$$\forall s' < s:\ \tilde{v}(\alpha)_{s'} = \tilde{v}(\beta)_{s'}.$$

Observe that the value of s is uniquely determined and this defines a well-defined and total order on Φ^+. If now $\alpha \prec \beta$ with $s \in S$ as above and $\gamma = \lambda\alpha + \mu\beta \in \Phi^+$, we get $\sigma(\gamma) = \lambda\sigma(\alpha) + \mu\sigma(\beta)$. In particular

$$\tilde{v}(\alpha)_s < \tilde{v}(\gamma)_s = \frac{\lambda\sigma(\alpha)\tilde{v}(\alpha)_s + \mu\sigma(\beta)\tilde{v}(\beta)_s}{\lambda\sigma(\alpha) + \mu\sigma(\beta)} < \tilde{v}(\beta)_s$$

and for each $s' < s$, we get

$$\tilde{v}(\alpha)_{s'} = \tilde{v}(\gamma)_{s'} = \frac{\lambda\sigma(\alpha)\tilde{v}(\alpha)_{s'} + \mu\sigma(\beta)\tilde{v}(\beta)_{s'}}{\lambda\sigma(\alpha) + \mu\sigma(\beta)} = \tilde{v}(\beta)_{s'}.$$

This verifies the reflection order property. □

For finite Coxeter groups, we have the following characterization due to Dyer [Dye93].

Lemma 137. *Let W be finite with longest element w_0 and let $\prec$ be a total order on Φ^+. Then the following are equivalent.*

(a) The order $\prec$ defines a reflection order on Φ^+.

(b) There exists a (unique) reduced word $w_0 = s_{\alpha_1}\cdots s_{\alpha_n}$ such that $\beta_1 \prec \cdots \prec \beta_n$, where

$$\beta_i = s_{\alpha_n}\cdots s_{\alpha_{i+1}}(\alpha_i).$$

Proof. (a) $\implies$ (b). Enumerate the positive roots as $\Phi^+ = \{\beta_1 \prec \cdots \prec \beta_n\}$ and define for $i = 1,\dots,n$ the root

$$\alpha_i = s_{\beta_n}\cdots s_{\beta_{i+1}}(\beta_i) \in \Phi.$$

We claim that all these roots are simple (in particular positive), via induction on $n-i$.

If $i \in \{1,\dots,n\}$ and the claim has been proved for all $i' > i$, note that

$$s_{\beta_n}\cdots s_{\beta_{i+1}} = s_{\alpha_{i+1}}\cdots s_{\alpha_n}.$$

Moreover, we have $1 < s_{\beta_n} < \cdots < s_{\beta_n} \cdots s_{\beta_{i+1}}$ such that the right one of the above words is reduced and

$$\mathrm{inv}(s_{\alpha_{i+1}} \cdots s_{\alpha_n}) = \{\beta_{i+1}, \ldots, \beta_n\}.$$

In particular, $\alpha_i = s_{\alpha_{i+1}} \cdots s_{\alpha_n}(\beta_i) \in \Phi^+$. If α_i was not simple, we could write $\alpha_i = \lambda_1\gamma_1 + \lambda_2\gamma_2$ for positive scalars $\lambda_1, \lambda_2 > 0$ and distinct positive roots $\gamma_1, \gamma_2 \in \Phi^+$ (e.g. by choosing a simple root which gets sent to a negative root under s_{α_i}). Put $\tilde{\gamma}_j = s_{\alpha_n} \cdots s_{\alpha_{i+1}}\gamma_j$, $j = 1, 2$, so that $\beta_i = \lambda_1\tilde{\gamma}_1 + \lambda_2\tilde{\gamma}_2$.

If $\tilde{\gamma}_j$ is positive, then the condition $\gamma_j = s_{\alpha_{i+1}} \cdots s_{\alpha_n}\tilde{\gamma}_j \in \Phi^+$ implies $\tilde{\gamma}_j \preceq \beta_i$. Similarly, if $\tilde{\gamma}_j \in \Phi^-$ then $-\tilde{\gamma}_j \succ \beta_i$.

In case both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are positive, they must both be $\preceq \beta_i$ and then the fact $\beta_i = \lambda_1\tilde{\gamma}_1 + \lambda_2\tilde{\gamma}_2$ contradicts (a).

If, say, $\tilde{\gamma}_1$ is positive and $\tilde{\gamma}_2$ is negative, we get

$$-\tilde{\gamma}_2 \succ \beta_i \succeq \tilde{\gamma}_1, \qquad \tilde{\gamma}_1 = \frac{1}{\lambda_1}\beta_i + \frac{\lambda_2}{\lambda_1}(-\tilde{\gamma}_2),$$

contradiction again.

Finally, if both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are negative, we get $\beta_i \in \Phi^-$ which is absurd. This finishes the induction.

The inductive proof shows moreover that

$$s_{\alpha_1} \cdots s_{\alpha_n}$$

is a reduced word of an element sending every root in Φ^+ to a negative root. This is the desired reduced word for (b).

(b) $\implies$ (a). If $i < j$ are indices such that $\beta := \lambda\beta_i + \mu\beta_j \in \Phi^+$, note that

$$\begin{aligned} \mathrm{inv}(s_{\alpha_i} \cdots s_{\alpha_n}) =& \{\beta_i, \ldots, \beta_n\} \\ \mathrm{inv}(s_{\alpha_{j+1}} \cdots s_{\alpha_n}) =& \{\beta_{j+1}, \ldots, \beta_n\}. \end{aligned}$$

Since both β_i and β_j are in $\mathrm{inv}(s_{\alpha_i} \cdots s_{\alpha_n})$, so must be β. Similarly, β cannot be in $\mathrm{inv}(s_{\alpha_{j+1}} \cdots s_{\alpha_n})$. Hence $\beta \in \{\beta_{j+1}, \ldots, \beta_{i-1}\}$. □

Using Matsumoto's theorem to pass between different reduced words for w_0, one can pass from one reflection order to another using a sequence of well-defined operations, cf. [BFP98]. A typical application of reflection orders is the following result, whose proof we will not state.

Theorem 138 ([BB05, Theorem 5.3.4]). *Let $\prec$ be a reflection order and $v, w \in W$. Let P be the set of all sequences $(\alpha_1, \ldots, \alpha_n)$ of positive roots such that*

- *$\alpha_1 \prec \cdots \prec \alpha_n$ and*
- *$v < vs_{\alpha_1} < \cdots < vs_{\alpha_1} \cdots s_{\alpha_n} = w$.*

Then the R-polynomial of v, w is given by

$$R_{v,w} = q^{(\ell(u)-\ell(v))/2} \sum_{(\alpha_1,\ldots,\alpha_n)\in P} (q^{1/2} - q^{-1/2})^n \in \mathbb{Z}[q]. \qquad \square$$

Exercise 139. List the reflection orders for S_3, S_4.

Exercise 140. Assume that W is a finite group with reflection order $\prec$ and that

$$\{\alpha_1 \prec \cdots \prec \alpha_n\} \subset \Phi^+$$

are some positive roots such that a linear combination

$$\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n = \alpha \in \Phi^+$$

is a positive root as well, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geqslant 0}$. Prove $\alpha_1 \preceq \alpha \preceq \alpha_n$.

21 Reflection subgroups

Proposition 141. *Let $W' \subseteq W$ be a* reflection subgroup*, i.e. such that $T' := W' \cap T$ generates W'. Define*

$$S' := \{t \in T' \mid \forall t \neq t' \in T' : \ \ell(tt') > \ell(t)\}.$$

Define $\Phi' \subseteq \Phi^+$ to be the set of all positive roots $\alpha \in \Phi^+$ such that $s_\alpha \in W'$ (so $s_\alpha \in T'$). Call a root $\alpha \in \Phi'$ simple for W' *if for any linear combination*

$$\alpha = \sum_{\beta\in\Phi'} \lambda_n\beta \in V, \quad \lambda_\beta \in \mathbb{R}_{\geqslant 0} \text{ almost all zero,}$$

we must have

$$\lambda_\beta = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases}$$

(a) (W', S') is a Coxeter group.

(b) A root $\alpha \in \Phi'$ is simple for W' if and only if $s_\alpha \in S'$. Under that identification, the geometric representation of W' is isomorphic to the action of W' on the subspace of V generated by the simple roots for W'.

Proof. W' **generated by** S': For this, it suffices to see that every element $t \in T'$ is a product of elements in S'. We show this via induction on $\ell_W(t)$. If $t \in S'$ (e.g. t is simple), there is not much to show.

Otherwise, we find $t' \in T'$ with $\ell_W(t't) < \ell_W(t)$. Pick a reduced word

$$t = s_{\alpha_1} \cdots s_{\alpha_\ell}$$

such that

$$\mathrm{inv}(t) = \{\alpha_\ell, s_{\alpha_\ell}(\alpha_{\ell-1}), \ldots, s_{\alpha_\ell} \cdots s_{\alpha_2}(\alpha_1)\}.$$

Writing $t = s_\alpha$, we see that α must appear in this list. If it appears in position i, then $t = s_{\alpha_1} \cdots s_{\alpha_i} \cdots s_{\alpha_1}$, which means (by reducedness) $i \geqslant (\ell-1)/2$. Considering $t = t^{-1}$, a similar argument shows $i \leqslant (\ell-1)/2$ such that $i = (\ell-1)/2$ and $t = s_{\alpha_1} \cdots s_{\alpha_i} \cdots s_{\alpha_1}$ is reduced.

We may and do assume that $\alpha_j = \alpha_{\ell+1-j}$ for all $j \in \{1, \ldots, \ell\}$. Write now $t' = s_\beta$. Up to replacing t' by $tt't$, we may assume that

$$\beta = s_{\alpha_1} \cdots s_{\alpha_{j-1}}(\alpha_j)$$

for some index $j \leqslant i = (\ell-1)/2$. If $j = i$, we would get $t = t'$, contradiction. Hence $j < i$ and

$$\begin{aligned} \ell_W(t') =& \ell_W(s_{\alpha_1} \cdots s_{\alpha_j} \cdots s_{\alpha_1}) \leqslant 2j-1 < \ell, \\ \ell_W(t'tt') =& \ell_W(s_{\alpha_1} \cdots \widehat{s_{\alpha_j}} \cdots s_{\alpha_{(\ell-1)/2}} \cdots \widehat{s_{\alpha_{\ell+1-j}}} \cdots s_{\alpha_\ell}) \leqslant \ell - 2. \end{aligned}$$

By induction, both t' and $t'tt'$ lie in the subgroup of W' generated by S'. Hence so does t, finishing the proof of W' being generated by S'.

Characterization of S'. For $\alpha \in \Phi'$, the condition $s_\alpha \in S'$ is equivalent to saying that for all $\alpha \neq \beta \in \Phi'$ we have $s_\alpha(\beta) \in \Phi^+$ (or equivalently in Φ').

Suppose first that $s_\alpha \in S'$. We have to show that α is simple for W'. So consider any linear combination

$$\alpha = \sum_{\beta \in \Phi'} \lambda_\beta \beta$$

as above. Apply s_α to write

$$-\alpha = \sum_{\beta \in \Phi'} \lambda_\beta s_\alpha(\beta) = -\lambda_\alpha \alpha + \sum_{\alpha \neq \beta \in \Phi'} \lambda_\beta s_\alpha(\beta).$$

So $(\lambda_\alpha - 1)\alpha$ is a $\mathbb{R}_{\geqslant 0}$-linear combination of positive roots, showing $\lambda_\alpha \geqslant 1$. Now

$$\sum_{\alpha \neq \beta \in \Phi'} \lambda_\beta \beta = (1 - \lambda_\alpha)\alpha \leqslant 0,$$

so that $\lambda_\beta = 0$ for all $\alpha \neq \beta \in \Phi'$. The claim follows.

Suppose now conversely that α is simple for W'. If we had $s_\alpha(\beta) \in \Phi^-$ for some $\alpha \neq \beta \in \Phi'$, we got

$$\alpha = \beta^\vee(\alpha)\beta - s_\alpha(\beta)$$

as a linear combination of two roots in Φ', both not equal to α. This contradict the condition of α being simple for W'.

Weak Exchange Condition. Let $w = t_1 \cdots t_n \in W'$ be reduced for (W', S') and $\alpha \in \Phi'$ with $\ell_{W'}(ws_\alpha) \leqslant \ell_{W'}(w)$.

If $w\alpha \leqslant 0$, we find an index i with

$$t_i \cdots t_n \alpha \leqslant 0, \quad t_{i+1} \cdots t_n \alpha \geqslant 0.$$

So $\beta = t_{i+1} \cdots t_n \alpha \in \Phi'$ satisfies $t_i \beta \leqslant 0$. Since $t_i \in S'$, the above argument shows $t_i = s\beta$ and we get the Weak Exchange Condition.

If $w\alpha \geqslant 0$, the same argument proves $\ell_{W'}(ws_\alpha) > \ell_{W'}(w)$.

Geometric Representation. Define coroots $\alpha^\vee : V \to \mathbb{R}$ for all $\alpha \in \Phi^+$ such that $s_\alpha(v) = v - \alpha^\vee(v)\alpha$. This yields a faithful representation of W' on the subspace of V spanned by Φ'. We can verify the defining properties of the geometric representation by noticing that each root in Φ' is a $\mathbb{R}_{\geqslant 0}$-linear combination of the simple roots for W'. $\square$

Definition 142. A *dihedral reflection subgroup* of W is a subgroup W' generated by two reflections.

Lemma 143. *Let $\prec$ be a total order on T. Then the following are equivalent:*

(a) We get a reflection order on Φ^+ defined by

$$\alpha \prec \beta :\Longleftrightarrow s_\alpha \prec s_\beta$$

(b) For every dihedral reflection subgroup $W' \subseteq W$ with canonical generators S' as in Proposition 141, we may write $S' = \{t, t'\}$ such that

$$t \prec t'tt' \prec {}^{tt'}t \prec \cdots \prec {}^{t't}t' \prec tt't \prec t'$$

Proof. We may certainly write $S' = \{s_\alpha, s_\beta\}$ for some positive roots α, β which are simple for W'. The reflections listed in (b) come from the roots

$$\alpha, s_\beta(\alpha) s_\alpha s_\beta(\alpha), \ldots, s_\beta s_\alpha(\beta), s_\alpha(\beta), \beta.$$

Carefully analysing the geometric representation on dihedral groups, we see that $\prec$ defines a reflection order on Φ' if and only if (b) is satisfied. Of course, if (a) is satisfied, then $\prec$ is a reflection order on Φ'.

Assume now that (b) is satisfied. Let $\alpha, \beta \in \Phi^+$ and $\lambda, \mu \in \mathbb{R}_{>0}$ such that $\lambda\alpha + \mu\beta \in \Phi^+$. Let W' be the group generated by s_α and s_β. Then $\prec$ being a reflection order on Φ' implies that $\alpha \prec \lambda\alpha + \mu\beta \prec \beta$ or vice versa. Hence (a) follows. $\square$

Exercise 144. Prove the claimed characterization of reflection orders on dihedral groups.

22 Demazure products

In this section, we specialize the Hecke algebra further to the case $q = 0$. Up to slightly optimizing on signs, we get the presentation

$$\begin{aligned} T_w T_{w'} &= T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w'), \\ T_s^2 &= T_s \text{ if } s \in S. \end{aligned}$$

Proposition 145. *Let $w_1, w_2 \in W$ be two elements. Then $T_{w_1} T_{w_2}$ has the form T_z for some $z \in W$. Each of the following three sets has a unique maximum with respect to Bruhat order, which is moreover equal to z.*

$$\begin{aligned} &\{w_1' w_2 \mid w_1' \leqslant w_1 \text{ and } \ell(w_1' w_2) = \ell(w_1') + \ell(w_2)\}, \\ &\{w_1 w_2' \mid w_2' \leqslant w_2 \text{ and } \ell(w_1 w_2') = \ell(w_1) + \ell(w_2')\}, \\ &\{w_1' w_2' \mid w_1' \leqslant w_1 \text{ and } w_2' \leqslant w_2\}. \end{aligned}$$

Proof. Pick a reduced word $w_2 = s_1 \cdots s_n$. Iterating the above relations n times, we see

$$T_{w_1} T_{w_2} = T_{w_1 s_{i_1} \cdots s_{i_k}}$$

for some indices $1 \leqslant i_1 < \cdots < i_k \leqslant n$. In particular, the element z as required exists, is uniquely determined and an element of the second and third set. A completely analogous argument shows that z also lies in the first set.

It remains to show that every element in the third set is $\leqslant z$ in the Bruhat order. We do this via induction on $\ell(w_2)$. If $w_2 = 1$, we get $z = w_1$ and the claim is clear. Otherwise, pick a simple reflection s with $w_2 s < w_2$, and let $T_{w_1} T_{w_2 s} = T_{z'}$. If $w_1' \leqslant w_1$ and $w_2' \leqslant w_2$, we get $\min(w_2', w_2' s) \leqslant w_2 s$ and hence

$$w_1' \min(w_2', w_2' s) \leqslant z' \leqslant z.$$

So focus on the case $w_2' s < w_2'$. Then $w_1' w_2' s \leqslant z$. Since $zs > z$, we get $w_1' w_2' \leqslant z$ as well. This finishes the induction and the proof. □

Definition 146. For $w_1, w_2 \in W$, the element z in the above proposition is called *Demazure product* of w_1, w_2 and denoted $w_1 * w_2$. Note that $(W, *)$ is a monoid but not a group.

Corollary 147. *In the general setting of a Hecke algebra (before the specializations), consider the product*

$$T_{w_1} T_{w_2} = \sum_{w_3 \in W} f_{w_3} T_{w_3}.$$

Then $f_{w_3} = 0$ unless $w_3 \leqslant w_1 * w_2$. We have

$$f_{w_1 * w_2} \equiv q_2^{\ell(w_1) + \ell(w_2) - \ell(w_1 * w_2)} \pmod{q_1}.$$

If $w_3 \neq w_1 * w_2$, then $f_{w_3} \equiv 0 \pmod{q_1}$.

Proof. Since the specializations $q_2 \to 1, q_1 \to 0$ yield the above 0-Hecke algebra, the congruence statements are easily verified. Use that by the definition of the Hecke algebra, f_{w_3} is always a sum of monomials $q_1^{n_1} q_2^{n_2}$ such that $\ell(w_1) + \ell(w_2) - \ell(w_3) = 2n_1 + n_2$.

From the definition of the Hecke algebra, it is clear that any w_3 with $f_{w_3} \neq 0$ must be of the form $w_3 = w_1 w_2'$ with $w_2' \leqslant w_2$. $\square$

We get similar statements when expanding $T_{w_1} C_{w_2}$ using the basis $C_\bullet$. One can show that the Kazhdan-Lusztig polynomials $P_{v,w}$ simplify to 1 under the specialization $q \to 0$.

Exercise 148. Show that if W is finite, then $w_0 * w = w * w_0 = w_0$ for all $w \in W$.

Exercise 149. Show that $(w_1 * w_2)^{-1} = w_2^{-1} * w_1^{-1}$.

23 Finiteness results

For this section, assume that S is a finite set.

Let us write

$$T_{w_1} T_{w_2} = \sum_{w_3 \in W} f_{w_3} T_{w_3}, \quad f_{w_3} \in \mathbb{Z}[q_1, q_2]$$

for a general Hecke algebra. It was an important conjecture of Lusztig, which has been proved now, that the q_2-degree of f_{w_3} is always bounded by a constant depending only on (W, S). In this section, we demonstrate the baby version claiming that

$$\sup\{\ell(w_1) + \ell(w_2) - \ell(w_1 * w_2) \mid w_1, w_2 \in W\} < +\infty.$$

Definition 150. For $w \in W$, we define the *cone type* of w to be

$$C(w) = \{w' \in W \mid \ell(ww') = \ell(w) + \ell(w')\} \subseteq W.$$

For individual $w \in W$, the cone type $C(w) \subseteq W$ may typically be an infinite set. It interesting to observe that many distinct group elements may have the same cone type. We even have the following striking result.

Theorem 151. *There are only finitely many different cone types for W, i.e. the set*

$$\{C(w) \mid w \in W\} \subseteq 2^W$$

is finite.

We obtain the following consequences for Demazure products.

Corollary 152. *For $w_1, w_2 \in W$, the value of*

$$w_1^{-1}(w_1 * w_2) w_2^{-1} \in W$$

depends only on the pair $(C(w_1), C(w_2^{-1})) \in 2^W \times 2^W$. Moreover,

$$0 \leqslant \ell(w_1) + \ell(w_2) - \ell(w_1 * w_2) \leqslant N$$

for some constant $N \in \mathbb{Z}$ depending only on (W, S).

Proof. If $C(w_1) = C(w_1')$ and $C(w_2^{-1}) = C((w_2')^{-1})$, we get

$$w_1^{-1}(w_1 * w_2) = \max(\tilde{w}_2 \leqslant w_2 \mid \tilde{w}_2 \in C(w_1)) = (w_1')^{-1}(w_1' * w_2).$$

A similar argument proves $(w_1' * w_2)w_2^{-1} = (w_1' * w_2')(w_2')^{-1}$. This shows the independence claim.

We calculate

$$\ell(w_1) + \ell(w_2) - \ell(w_1 * w_2) = \ell(w_1) - \ell\Big((w_1 * w_2)w_2^{-1}\Big) \leqslant \ell\Big(w_1^{-1}(w_1 * w_2)w_2^{-1}\Big).$$

There are only finitely many possibilities for the right-hand side. □

The proof of Theorem 151 is rather involved, cf. [BB05, Chapter 4]. We can only outline the major steps.

Definition 153. Let $\alpha, \beta \in \Phi^+$. We say that α *dominates* β if any $w \in W$ such that $w\alpha \in \Phi^-$ satisfies $w\beta \in \Phi^-$.

We say that α is *humble* if it only dominates itself and not other root.

Observe that simple roots are humble.

Lemma 154. *Let $\alpha \in \Phi^+$ be humble and $s \in S$ such that $\alpha \neq \alpha_s$. Then $s\alpha$ is* not *humble if and only if the following conditions are both satisfied.*

- $\alpha_s^\vee(\alpha) < 0$, *or equivalently $s\alpha \in \alpha + \mathbb{R}_{>0}\alpha_s$ and*
- *The group generated by s, s_α is infinite, or equivalently*

$$\alpha_s^\vee(\alpha)\alpha^\vee(\alpha_s) \geqslant 4.$$

Proof. First assume that both stated conditions are satisfied. We claim that $s\alpha$ dominates α_s. Indeed, if this was not the case, we could find some $w \in W$ with $ws\alpha \in \Phi^-$ and $w\alpha_s \in \Phi^+$. Then $\alpha, \alpha_s \in \mathrm{inv}(ws)$, so the infinitely many roots of the form $\lambda\alpha + \mu\alpha_s, \lambda, \mu > 0$ are also in $\mathrm{inv}(ws)$. This contradicts ws having finite length.

Now assume conversely that $s\alpha$ is not humble. Then it dominates a root β. If $\beta \neq \alpha_s$, then $s\beta \in \Phi^+$ and α dominates $s\beta$, contradiction. Hence $s\alpha$ dominates α_s and no other root. Certainly, $s\alpha$ would dominate every positive linear combination of α_s and $s\alpha$, so α_s must be the only such linear combination. In particular, $\alpha \notin \mathbb{R}_{>0}s\alpha + \mathbb{R}_{>0}\alpha_s$, showing the first condition. The second one follows from analysing the dihedral group generated by s and s_α. If it is finite, we would find an element w in that group with $w\alpha_s \in \Phi^+$ and $ws\alpha \in \Phi^-$. □

Corollary 155. *For $w \in W$, denote the set of* humble inversions *by*

$$\mathrm{huminv}(w) = \{\beta \in \mathrm{inv}(w) \mid \beta \textit{ is humble}\}.$$

If $s \in S$ satisfies $ws > w$, then

$$\mathrm{huminv}(ws) = \{\alpha_s\} \cup \{s\beta \mid \beta \in \mathrm{huminv}(w) \textit{ and } s\beta \textit{ is humble.}\}.$$

Proof. Since $\mathrm{inv}(ws) = \{\alpha_s\} \cup s\,\mathrm{inv}(w)$, the inclusion $\subseteq$ is clear. For the reverse inclusion, pick $\alpha_s \neq \beta \in \mathrm{huminv}(ws)$. Then $s\beta \in \mathrm{inv}(w)$ and β is humble. We have to show that $s\beta$ is humble as well. If it was not humble, we would get infinitely many roots of the form $\lambda\beta + \mu\alpha_s \in \mathrm{inv}(ws)$ for $\lambda, \mu > 0$, which is impossible. □

If $w, w' \in W$ and $w' = s_1 \cdots s_n$ is a reduced word, we have

$$w' \in C(w) \iff \forall i:\ \ell(ws_1 \cdots s_i) > \ell(ws_1 \cdots s_{i-1}) \iff \forall i:\ \alpha_{s_i} \notin \mathrm{huminv}(ws_1 \cdots s_{i-1}).$$

Using the above corollary, the last condition only depends on the set $\mathrm{huminv}(w) \subseteq \Phi^+$. So Theorem 151 follows from the following result.

Proposition 156. *There are only finitely many humble roots.*

For the proof of the proposition, observe that if α is a non-simple humble root, we can find a simple root α_s with $s_\alpha s < s_\alpha$, or equivalently $s\alpha \in \alpha + \mathbb{R}_{<0}\alpha_s$. Then $s\alpha$ is humble as well and we get a sequence

$$\alpha = \alpha_1 \xrightarrow{s_1} \alpha_2 \xrightarrow{s_2} \cdots \xrightarrow{s_{n-1}} \alpha_n$$

of humble roots ending in a simple root α_n. If there are infinitely many humble roots, there exist such sequences of arbitrary length.

One has to develop the theory of humble roots a lot further to associate some invariants for which there are only finitely many possibilities and which are, in a sense, monotonic with respect to such sequences. If such sequences are longer than the number of invariants, we would get adjacent humble roots $\alpha \xrightarrow{s} \beta$ with identical invariants, and can use this to derive a contradiction.

Exercise 157. Show that if W is finite, then the number of distinct cone types is equal to the number of elements in W. Moreover, show that every root in a finite group is humble.

Exercise 158. Show that if W is infinite, then each cone type $C(w)$ is an infinite subset of W.

Exercise 159. Give an example of a Coxeter group (W, S) and two distinct elements $w_1, w_2 \in W$ such that $C(w_1) = C(w_2)$.

24 Conjugacy classes

The structure of conjugacy classes in Coxeter groups is of central interest.

Definition 160. Let $\mathcal{O} \subseteq W$ be a conjugacy class and $w \in W$.

(a) We write $\ell(\mathcal{O}) := \min\{\ell(w) \mid w \in \mathcal{O}\}$ and $\mathcal{O}_{\min}$ for the elements in $\mathcal{O}$ of length $\ell(\mathcal{O})$.

(b) If $s \in S$ and $\ell(sws) \leqslant \ell(w)$, we call sws an *elementary cyclic shift* of w and write $w \xrightarrow{s} sws$ (not to be confused with the similar, but incompatible notation used for

Bruhat order). We call $w' \in W$ a *cyclic shift* of w if there is a sequence of elementary cyclic shifts

$$w = w_1 \xrightarrow{s_1} w_2 \xrightarrow{s_2} \cdots \xrightarrow{s_n} w_{n+1} = w'.$$

We write $w \to w'$ to indicate this relation.

(c) If $v \in W$ satisfies $\ell(wv) = \ell(w)+\ell(v)$ and $\ell(v^{-1}wv) = \ell(w)$, we call $v^{-1}wv$ *elementary strongly conjugate* to w. We say that w and $w' \in W$ are *strongly conjugate* if there is a sequence

$$w = w_1, \dots, w_n = w'$$

such that w_i and w_{i+1} are strongly conjugate.

Observe that if $w \to w'$ and $\ell(w) = \ell(w')$, then also $w' \to w$ and w' is strongly conjugate to w.

Theorem 161 ([Mar18]). *Let $w \in W$ and $\mathcal{O} \subseteq W$ its conjugacy class.*

(a) There is an element $w' \in \mathcal{O}_{\min}$ such that $w \to w'$.

(b) If $w, w' \in \mathcal{O}_{\min}$, then they are strongly conjugate. □

Marquis' proof is rather subtle, generalizing previous works of Geck-Pfeiffer for finite Coxeter groups and He-Nie for affine ones.

Example 162. In the group S_n, the conjugacy class of an element w written in cycle notation

$$w = (i_{1,1}, \dots, i_{1,c(1)})(i_{2,1}, \dots, i_{2,c(2)}) \cdots (i_{n,1}, \dots, i_{n,c(n)})$$

with $c(1) \geqslant \cdots \geqslant c(n)$ is determined by the vector $(c(1), \dots, c(n))$, up to adding and removing trivial cycles. A typical minimal length element is given by

$$(1, 2, 3)(5, 6)(10, 11, 12, 13).$$

The conjugacy classes in S_3 are given by

$$\{1\}, \{(1\ 2), (2\ 3), (1\ 3)\}, \{(1\ 2\ 3), (1\ 3\ 2)\}.$$

Note that the second class $\mathcal{O}$ has the simple reflections as minimal length elements $\mathcal{O}_{\min} = \{s_1, s_2\}$. Certainly we do *not* have $s_1 \to s_2$. However, the element $v = s_2 s_1$ satisfies $s_1 v = w_0$, which has length 3, and $v^{-1} w_0 = s_2$.

Definition 163. Let $\mathcal{H} = \mathcal{H}(W)$ be the Iwahori-Hecke algebra, defined over some ring A and some $q_1, q_2 \in A$, such that $q_1 \in A^\times$. The *commutator* $[\mathcal{H}, \mathcal{H}]$ of $\mathcal{H}$ is the A-module generated by

$$[h_1, h_2] = h_1 h_2 - h_2 h_1, \qquad h_1, h_2 \in \mathcal{H}.$$

The *cocenter* is the quotient A-module $\mathcal{H}/[\mathcal{H}, \mathcal{H}]$.

Lemma 164. *(a) If $w_1, w_2 \in W$ are strongly conjugate, then the images of T_{w_1} and T_{w_2} in the cocenter $\mathcal{H}/[\mathcal{H},\mathcal{H}]$ coincide.*

(b) The cocenter $\mathcal{H}/[\mathcal{H},\mathcal{H}]$ is generated, as A-module, by the images of T_w where $w \in W$ runs through all elements which are of minimal length in their conjugacy class.

Proof. (a) It suffices to prove the claim for elementary strongly conjugate elements. If $v \in W$ satisfies $\ell(w_1 v) = \ell(w_1) + \ell(v), \ell(v w_2) = \ell(v) + \ell(w_2)$ and $w_1 v = v w_2$, we get

$$T_{w_1} = T_{w_1 v} T_v^{-1} \equiv T_v^{-1} T_{w_1 v} = T_v^{-1} T_{v w_2} = T_{w_2} \pmod{[\mathcal{H},\mathcal{H}]}.$$

(b) Induction on $\ell(w)$. If w has minimal length in its σ-conjugacy class, we are done. Otherwise, we find by Marquis' theorem a sequence

$$w = w_1 \xrightarrow{s_1} \cdots \xrightarrow{s_n} w_{n+1}$$

with $\ell(w_1) = \cdots = \ell(w_n) > \ell(w_{n+1})$. By (a), we get $T_w \equiv T_{w_n} \pmod{[\mathcal{H},\mathcal{H}]}$. We compute

$$\begin{aligned} T_{w_n} =& T_{s_n w_{n+1} s_n} = T_{s_n} T_{w_{n+1}} T_{s_n} \\ \equiv& T_{s_n}^2 T_{w_{n+1}} = (q_2 T_{s_n} + q_1) T_{w_{n+1}} = q_2 T_{s_n w_{n+1}} + q_1 T_{w_{n+1}} \pmod{[\mathcal{H},\mathcal{H}]}. \end{aligned}$$

Since $\ell(w_{n+1}), \ell(s w_{n+1}) < \ell(w)$, we are done by induction. □

Definition 165. For a conjugacy class $\mathcal{O} \subseteq W$, write $T_{\mathcal{O}} \in \mathcal{H}/[\mathcal{H},\mathcal{H}]$ for the image of T_w for any $w \in \mathcal{O}_{\min}$.

This is well-defined by Marquis' theorem. We saw above that these generate $\mathcal{H}/[\mathcal{H},\mathcal{H}]$ as A-module.

Theorem 166 ([HN14])**.** *Suppose that (W, S) is spherical or affine. Then $\mathcal{H}/[\mathcal{H},\mathcal{H}]$ is a free A-module with basis*

$$\{T_{\mathcal{O}} \mid \mathcal{O} \subseteq W \text{ conjugacy class}\}. \qquad \square$$

It is an open problem whether the theorem holds for arbitrary Coxeter groups. But at least, we have the following map:

Definition 167. Define

$$\tau : \mathcal{H} \to A,$$

to be the A-linear map sending T_w for $w \in W$ to

$$\tau(T_w) = \begin{cases} 1, & w = 1, \\ 0, & w \neq 1. \end{cases}$$

Lemma 168. *We have $\tau(hh') = \tau(h'h)$ for all $h, h' \in A$. In other words, we get a well-defined map*

$$\mathcal{H}/[\mathcal{H},\mathcal{H}], \quad h + [\mathcal{H},\mathcal{H}] \mapsto h.$$

Proof. Observe that

$$\tau(T_{w_1}T_{w_2}) = \begin{cases} q_1^{\ell(w_1)}, & w_1 = w_2^{-1}, \\ 0, & w_1 \neq w_2^{-1}. \end{cases}$$

Then both statements follow immediately. □

Definition 169. If the statement of Theorem 166 holds true, we define the *class polynomials* $f_{w,\mathcal{O}} \in A$ for $w \in W$ and $\mathcal{O} \subseteq W$ via the identity

$$T_w + [\mathcal{H},\mathcal{H}] = \sum_{\mathcal{O}\subseteq W} f_{w,\mathcal{O}} T_{\mathcal{O}} \in \mathcal{H}/[\mathcal{H},\mathcal{H}].$$

Corollary 170. *If W is spherical, the center of $\mathcal{H}$ is a free A-module with basis*

$$z_{\mathcal{O}} := \sum_{w\in W} f_{w,\mathcal{O}} T_{w^{-1}} q_1^{-\ell(w)}$$

indexed by the conjugacy classes $\mathcal{O} \subseteq W$.

Proof. Define the scalar product

$$\mathcal{H}\times\mathcal{H} \to A, \qquad (h,h') \mapsto \tau(hh').$$

Let $h \in \mathcal{H}$. We have

$$\begin{aligned} & h \text{ central in } \mathcal{H} \\ \iff & hh' = h'h \text{ for all } h' \in \mathcal{H} \\ \iff & \tau(hh'T_{w^{-1}}) = \tau(T_{w^{-1}}h'h) \text{ for all } h' \in \mathcal{H}, w \in W \\ \iff & \tau(hh'h'') = \tau(h''h'h) \text{ for all } h', h'' \in \mathcal{H} \\ \iff & \tau(h\tilde{h}) = 0 \text{ for all } \tilde{h} \in [\mathcal{H},\mathcal{H}]. \end{aligned}$$

So the center of $\mathcal{H}$ is the orthogonal complement of $[\mathcal{H},\mathcal{H}]$ under this scalar product. Thus, the composition

$$Z(\mathcal{H}) \to \mathcal{H} \to \mathcal{H}/[\mathcal{H},\mathcal{H}]$$

yields an isomorphism of A-modules.

Let now $h \in \mathcal{H}$ by central. The above calculation shows that

$$\mathcal{H}/[\mathcal{H},\mathcal{H}] \to A, \qquad h' + [\mathcal{H},\mathcal{H}] \mapsto \tau(hh')$$

is well-defined. For $w \in W$, we compute

$$\begin{aligned}\tau(hT_w) =& \tau(h(T_w + [\mathcal{H}, \mathcal{H}])) = \tau\Big(h \textstyle\sum_{\mathcal{O}} f_{w,\mathcal{O}} T_{\mathcal{O}}\Big) \\ &= \sum_{\mathcal{O} \subseteq W} f_{w,\mathcal{O}} \tau(hT_{\mathcal{O}}) = \sum_{\mathcal{O} \subseteq W} \tau(z_{\mathcal{O}} T_w) \tau(hT_{\mathcal{O}}) = \tau\Big(\textstyle\sum_{\mathcal{O}} z_{\mathcal{O}} \tau(hT_{\mathcal{O}}) T_w\Big).\end{aligned}$$

Thus

$$h = \sum_{\mathcal{O} \subseteq W} \tau(hT_{\mathcal{O}}) z_{\mathcal{O}}.$$

So the center of $\mathcal{H}$, which is free with rank equal to the number of conjugacy classes in W, is contained in the submodule of $\mathcal{H}$ generated by the elements $z_{\mathcal{O}}$. This is only possible if the center of $\mathcal{H}$ is equal to that submodule and the $z_{\mathcal{O}}$ are linearly independent. □

A peculiar class of elements, which plays an important role in different aspects of Coxeter group theory, are the so-called *straight* elements.

Definition 171. An element $w \in W$ is called *straight* if for any $n \geqslant 1$, we have

$$\ell(w^n) = n\ell(w).$$

Lemma 172. *Let $w \in W$ and $C \in \mathbb{R}$ such that for all $n \geqslant 1$, we have*

$$\ell(w^n) \geqslant n\ell(w) - C.$$

(a) w is straight.

(b) Let $\mathcal{O}$ be the conjugacy class of w. Then $\mathcal{O}_{\min}$ is given precisely by the straight elements in $\mathcal{O}$.

Proof. (a) For $n, m > 0$, we get

$$m\ell(w^n) \geqslant \ell(w^{mn}) \geqslant mn\ell(w) - C.$$

Divide by m and take the limit $m \to \infty$.

(b) If $w' = v^{-1}wv$, then

$$\ell((w')^n) = \ell(v^{-1}w^n v) \in [\ell(w^n) - 2\ell(v), \ell(w^n) + 2\ell(v)].$$

If $\ell(w') \leqslant \ell(w) - 1$, we would get

$$\ell((w')^n) \leqslant n\ell(w') \leqslant n\ell(w) - n = \ell(w^n) - n,$$

contradicting the above estimate. Hence $w \in \mathcal{O}_{\min}$ and all straight elements in $\mathcal{O}$ are in $\mathcal{O}_{\min}$. If conversely $w' \in \mathcal{O}_{\min}$, we get $\ell(w') = \ell(w)$ so the above estimate shows

$$\ell((w')^n) \geqslant n\ell(w') - 2\ell(v).$$

Hence w' is straight by (a). □

Theorem 173 ([Mar18]). *Let $\mathcal{O}$ be a straight conjugacy class, i.e. containing a straight element. Then for any $w_1, w_2 \in \mathcal{O}_{\min}$, we have $w_1 \to w_2$.* □

So straight elements are nice, but how do we find them?

Proposition 174. *Let $w \in W$ and consider the iterated Demazure products*

$$w^{*,n} := w * w * w * \cdots * w \in W.$$

Define $w_n := (w^{*,n-1})^{-1} w^{*,n} \in W$.

(a) The sequence $(w_n)_{n \geqslant 1}$ stabilizes, i.e. there is an index N such that $w_N = w_n$ for all $n \geqslant N$.

(b) The element $w_\infty := w_N$ as in (a) is straight.

(c) If w' is strongly conjugate to w, then $(w')_\infty$ is conjugate to w_∞.

Proof. (a) By definition of the Demazure product, observe that the value of w_n only depends on w and the cone type $C(w^{*,n-1})$. Moreover, we have $w^{*,n-1} \leqslant^R w^{*,n}$ in the right weak order. Hence

$$C(w^{*,1}) \supseteq C(w^{*,2}) \supseteq \cdots .$$

By finiteness of cone types, this sequence stabilizes (if S is infinite, consider the parabolic subgroup generated by the support of w, which is always finite). Hence the sequence (w_n) stabilizes.

(b) By definition of the Demazure product, we get for every $n \geqslant 1$ that

$$\ell(w^{*,n+1}) = \ell(w^{*,n}) + \ell(w_n).$$

Thus

$$n\ell(w_N) = \ell(w^{*,N+n}) - \ell(w^{*,N}) \leqslant \ell\Big((w^{*,N})^{-1} w^{*,N+n}\Big) = \ell(w_N^n).$$

(c) It suffices to show this for elementary strongly conjugate elements. So let $v \in W$ such that

$$\ell(wv) = \ell(w) + \ell(v),\ \ell(vw') = \ell(v) + \ell(w') \text{ and } vw' = wv.$$

For $n \geqslant 1$, write

$$w^{*,n} * v = w^{*,n} v_n, \qquad v * (w')^{*,n} = v'_n (w')^{*,n}.$$

Similar to (a), the sequences $(v_n)_{n \geqslant 1}$ and $(v'_n)_{n \geqslant 1}$ stabilize. Observe that by associativity of the Demazure product, we get

$$w^{*,n} * v = w^{*,n-1} * w * v = w^{*,n-1} * (wv) = w^{*,n-1}(vw') = w^{*,n-1} * v * w'.$$

Repeating this argument, we get

$$w^{*,n} * v = v * (w')^{*,n}.$$

Now

$$\begin{aligned} v_{n-1} w_n v_n =& \Big[w^{*,n-1} * v\Big]^{-1} \Big[w^{*,n} * v\Big] = \Big[v * (w')^{*,n-1}\Big]^{-1} \Big[v * (w')^{*,n}\Big] \\ =& \Big[(w')^{*,n-1}\Big]^{-1} (v')^{-1}_{n-1} v'_n (w')^{*,n}. \end{aligned}$$

For sufficiently large n, we get $w_n = w_\infty$, $v_{n-1} = v_n$, $v'_{n-1} = v'_n$ and

$$\Big[(w')^{*,n-1}\Big]^{-1} (w')^{*,n} = w'_\infty.$$

We conclude

$$v_n^{-1} w_\infty v_n = w'_\infty.$$

This finishes the proof. □

This proposition allows to associate to every σ-conjugacy class $\mathcal{O} \subseteq W$ a corresponding straight σ-conjugacy class.

Exercise 175. Assume that S is finite. Show that

$$\sup\{\ell(w) - \ell(w_\infty) \mid w \in W\} < +\infty.$$

Conclude that there are infinitely many straight conjugacy classes if W is infinite and irreducible. Show that there is only one straight conjugacy class if W is finite.

Exercise 176. For $W = S_3$ and $A = \mathbb{Z}[q_1, q_2]$, write the image of T_{w_0} in $\mathcal{H}/[\mathcal{H}, \mathcal{H}]$ as A-linear combination of $T_{\mathcal{O}}$'s.

Suggestions for presentation topics

Those who formally enrolled for the course are required to give a short presentation of 20–30 minutes on a subject related to Coxeter groups. Longer talks are possible, but please inform me beforehand and keep it under 60 minutes.

Please try to give a motivating and instructive talk, focussing on results and applications rather than elaborate technical proofs. You can find a couple of suggestions for topics below, or you use your own topic after confirming it is suitable for this course. Please email me once you made a decision. Topics are first come, first serve.

The type D_n

The remaining infinite family of finite Coxeter groups is the type D_n for $n \geqslant 4$. Explain how these groups look like, how to do computations like length, descent sets and Bruhat order. You may follow [BB05, Section 8.2], or you construct the root system of type D_n (using any standard Lie theory reference) and define it using the usual Weyl group construction.

Order automorphisms

A *right weak order automorphism* is a bijective map $f : W \to W$ such that $v \leqslant_R w$ if and only if $f(v) \leqslant_R f(w)$. A *Coxeter group automorphism* is a group automorphism $f : W \to W$ with $f(S) = S$. Show that, in most of the interesting cases, these two notions agree following [BB05, Theorem 3.2.5]. Explain the situation for Bruhat order automorphisms, using [BB05, Theorem 2.3.5].

Finite reflection groups

A finite subgroup $G \leqslant \mathrm{GL}_n(\mathbb{R})$ generated by reflections is called a *finite reflection group*. It is a classical result that these are "the same as" finite Coxeter groups. Explain what this means, e.g. following [Hum90]. You may either sketch what goes into the proof of such a correspondence, or how to use a classification of finite Coxeter groups to obtain a classification of regular polyhedrons (e.g. Platonic solids).

Braid groups and Jones polynomial

In a seminal paper, Jones [Jon87] introduced the so-called *Jones polynomial* to knot theory, the study of smooth embeddings of the circle into $\mathbb{R}^3$. Following his paper, explain what Braid groups are, how they relate to Coxeter groups of type A_n and how they relate to knot theory. Explain how to compute the Jones polynomial of a knot and, as time permits, how Jones derives his polynomial from the representation theory of Iwahori-Hecke algebras.

Flag varieties

Let $V = \mathbb{C}^n$. A complete flag in V is a chain of sub-vector spaces

$$\{0\} \subsetneq U_1 \subsetneq \cdots \subsetneq U_{n-1} \subsetneq V,$$

so that $\dim U_i = i$. Let $G = \mathrm{GL}_n(\mathbb{C})$ and $B \subset G$ the subset of upper triangular matrices. Explain the one-to-one correspondence between complete flags in V and the *flag variety* G/B, e.g. following [Bri04].

Explain why the flag variety is *projective*, i.e. embeds into some projective space $\mathbb{P}^N(\mathbb{C})$ as closed subset (e.g. by embedding G/B into a suitable product of *Grassmannians*, then using projectivity of Grassmannians as a black box).

Define the Schubert cell $BwB \subset G/B$ for $w \in S_n$ and show why it is isomorphic (as variety, or manifold) to $\mathbb{C}^{\ell(w)}$. Explain some applications of flag varieties, or explain how to show that the covering relations of Schubert cells are given by the Bruhat order.

Further topics

- Shellability of Bruhat order [BB05, Section 2.7].
- Normal forms of reduced words [BB05, Section 3.4]. (taken)

- Introduction to Coxeter group computations with computer algebra (GAP, SAGE etc.).
- Young tableaux and the symmetric group.
- Unipotent orbits and conjugacy classes. (taken)
- Coxeter matroids. (taken)
- Categorification of Kazhdan-Lusztig polynomials. (taken)
- Conjugacy classes in finite Weyl groups. (taken)

References

[BB05] A. Björner and F. Brenti. *Combinatorics of Coxeter Groups*. Springer-Verlag Berlin Heidelberg, 2005.

[Bon17] C. Bonnafé. *Kazhdan-Lusztig Cells with Unequal Parameters*. Cham, Switzerland: Springer International Publishing, 2017.

[BFP98] F. Brenti, S. Fomin, and A. Postnikov. *Mixed Bruhat Operators and Yang-Baxter Equations for Weyl Groups*. International Mathematics Research Notices 8 (1998), pp. 419–441.

[Bri04] M. Brion. *Lectures on the geometry of flag varieties*. arXiv (2004). eprint: `math/0410240`.

[Deo77] V. V. Deodhar. *Some Characterizations of Bruhat Ordering on a Coxeter Group and Determination of the Relative Möbius Function*. Inventiones Mathematicae 39 (1977), p. 187.

[Deo89] V. V. Deodhar. *A note on subgroups generated by reflections in Coxeter groups*. Arch. Math. (Basel) (1989).

[Dye93] M. J. Dyer. *Hecke algebras and shellings of Bruhat intervals*. Compositio Mathematica 89.1 (1993), pp. 91–115.

[HN14] X. He and S. Nie. *Minimal length elements of extended affine Weyl groups*. Compositio Mathematica 150.11 (2014), pp. 1903–1927.

[Hum90] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge, England, UK: Cambridge University Press, 1990.

[Jon87] V. F. R. Jones. *Hecke Algebra Representations of Braid Groups and Link Polynomials on JSTOR*. Ann. Of Math. 126.2 (1987), pp. 335–388.

[KL79] D. Kazhdan and G. Lusztig. *Representations of Coxeter groups and Hecke algebras*. Invent. Math. 53.2 (1979), pp. 165–184.

[Mar18] T. Marquis. *Cyclically reduced elements in Coxeter groups*. arXiv (2018). eprint: `1812.02543`.

[Mat64] H. Matsumoto. *Générateurs et relations des groupes de Weyl généralisés*. Comptes Rendus Hebodomadaires Des Seances De L'Academie Des Sciences 258.13 (1964), p. 3419.