Carter's Conjugacy Classes in the Weyl Group

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@ Haiyu CHEN

A <u>Weyl group</u> W of complex simple Lie algebra can be regarded as the group of orthogonal transformations of Euclidean space V generated by the reflections $s_\alpha : \alpha \in \Phi$, where $V = \mathrm{span}\,\Phi$.
Each element $w \in W$ can be expressed in the form

$$w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}, \quad \alpha_i \in \Phi. \text{ (not necc. simple roots).}$$

with a <u>length function</u> $\ell(w)$ = smallest value k in any such expressions
= # eigenvalues of $w \curvearrowright V$ not equal to 1.

An expression $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ is <u>reduced</u> if $\ell(w) = k$.

$\Leftrightarrow$ $\alpha_1, \cdots, \alpha_k \in \Phi$ are linearly independent.

If $\alpha_1, \cdots, \alpha_k$ are mutually orthogonal, the reflections $s_{\alpha_1}, \cdots, s_{\alpha_k}$ commutes, so their products are involutions. In fact, every involution can be expressed as a product of $\ell(w)$ reflections corresponding to mutually orthogonal roots.

Def 1. Define $W_0 \subseteq W$ be a subset of elements $w \in W$ s.t. $w = w_1 w_2$ for some involutions $w_1^2 = w_2^2 = 1$ and

$$V_{-1}(w_1) \cap V_{-1}(w_2) = 0. \qquad V_{-1} \text{ is the eigenspace of } -1.$$

(It will turn out at the end that $W_0 = W$).

Prop 1. W_0 is a union of conjugacy classes of W.

proof sketch. $V_{-1}(w' w w'^{-1}) = w' V_{-1}(w)$.

Def 2. Let $w \in W_0$, and $w = w_1 w_2$ as in def 1. Write

$$w_1 = s_{\alpha_1} \cdots s_{\alpha_k}, \quad w_2 = s_{\alpha_{k+1}} \cdots s_{\alpha_{k+h}} \qquad k+h = \ell(w).$$

as product of reflections of mutually orthogonal roots.

Fix any such expression of w, we define a <u>graph Γ</u>, with one <u>node</u> corresponding to each $\alpha_1, \cdots, \alpha_{k+h}$. The nodes corresponding to distinct roots α, β are joined by a <u>bond</u> of strength $n_{\alpha\beta} \cdot n_{\beta\alpha}$, where

$$n_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \qquad n_{\beta\alpha} = \frac{2(\beta,r)}{(\beta,\beta)}$$

This product can only be 0, 1, 2, 3.

Prop 2. Γ is invariant under conjugation.

proof sketch. If $w = s_{\alpha_1} \cdots s_{\alpha_{k\text{th}}}$, $w' w w'^{-1} = s_{w'(\alpha_1)} \cdots s_{w'(\alpha_{k\text{th}})}$. Also $n_{\alpha\beta} = n_{w'(\alpha) w'(\beta)}$.

In this way, we may associate a graph to a conjugacy class of W inside W_0. Note that there can be more than one graph depending on the choice of roots in the expression.

Eg. 1). The conjugacy class of unit element is associated with $\Gamma = \phi$.

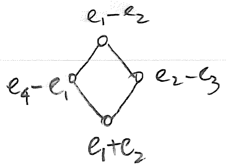
2). The conjugacy class of coxeter elements is associated with Γ, where Γ can be chosen to be the Dynkin diagram. (Coxeter elements are product of reflections corresp. to a complete set of simple roots in Φ. Coxeter shown that they form a single conjugacy class irrespective with simple roots chosen or the order of reflections they occur.) Coxeter elements are in W_0 because it is possible to divide nodes in any Dynkin diagram to 2 disjoint subsets of mutually orthogonal roots (bipartite).

Def. 3. An admissible diagram is a graph satisfy the following conditions.

(a). The nodes corresponds to a set of linearly independent roots.

(b) Each subgraph of Γ which is a cycle contains an even number of nodes.

The graph Γ associated with a conjugacy class of W inside W_0 is an admissible diagram.

Eg. 

$\Phi_D^+ = \{ e_i \pm e_j \mid i<j \}$.

Prop. 3. Every admissible diagram without cycles associated to a conjugacy class of W is the Dynkin diagram of some Weyl subgp of W.

proof sketch. A diagram corresp. to a set of lin. indep roots which are mutually obtuse is a Dynkin diagram (obtuse: $(\alpha,\beta) \le 0$)

The graphs which are Dynkin diagram of Weyl subgroups may be obtained by a standard algorithm:

1). Add a node corresponding to the negative of highest root to the Dynkin diagram. This is called extended Dynkin diagram. (Bourbaki appendix has all types)

2). Remove one or more nodes in all possible ways from the extended Dynkin diagram.

3). Take duals of the diagrams obtained in 2). from dual root system, meaning exchanging long and short roots.

4). Repeat process 1)-3). any number of times.

Ref. Borel and de Siebenthal [1] and independently, Dynkin [3].

Prop 4. Let Γ be an admissible diagram for Φ. Then there exists an admissible diagram $\bar{\Gamma}$ without cycles, with connected components $\bar{\Gamma}_i$ Dynkin diagram of Weyl group W_i. Then Γ can be obtained from $\bar{\Gamma}$ by replacing certain $\bar{\Gamma}_i$ with cycles associated with W_i, but not proper Weyl subgp of W_i.

proof sketch. For each connected component of Γ, take the smallest root system $\Phi' \subseteq \Phi$ containing roots of all nodes. No subsystem of Φ' contain those roots by minimality, so no proper Weyl subgp.

Prop 5. The type of W is uniquely determined by Γ if Γ is an admissible diagram for W but for no Weyl subgp of W.

proof sketch. Just consider $\Phi' = W_S(S)$, where S is the set of roots of Γ, W_S the gp generated by reflections of elements in S. $\Phi' = \Phi(W)$ by no proper subgp assumption if it is a root system. But this can be checked.

Classification of Admissible Diagrams

Theorem 6. Let Γ be an admissible diagram associated with indecomposible root system Φ but no proper subsystem, and Γ contains a cycle. Then Γ is one of the graphs in Table 2 P10-11. More over, these graphs have the properties described.

proof sketch. Case-by-Case discussion. Mainly use following facts:

- Every subgraph of an admissible diagram is an admissible diagram.
- Every subgraph without cycles is a Dynkin diagram.

One may obtain all admissible diagrams associated with an indecomposible root system by prop 4 and theorem 6. The algorithm in P2 gives all subgraphs with no cycles, then we may replace some connected component by one of the appropriate diagrams in table 2 with cycles.

Characteristic Polynomials

Theorem 7. Let $w \in W_0$ with associated admissible diagram Γ. Then the characteristic polynomial of $w \curvearrowright V$ is determined by Γ.

The characteristic polynomials determined by a connected admissible diagram are classified in Table 3. P23.

proof sketch. Take the roots of nodes in Γ as basis in a subspace U it spans. Then the characteristic polynomial on V is that on U times $(t-1)^{\dim V - \dim U}$. Long roots/short roots won't matter as they doesn't affect reflections.

Conjugacy Class and Admissible Diagrams

For classical types, we use the known description of Weyl group in terms of cycle types.

Theorem 8. (type A) Let W be a Weyl group of type A_ℓ. There is a one-to-one correspondence

$$\{\text{conjugacy classes of } W\} \leftrightarrow \left\{ \begin{array}{l} \text{admissible diagrams of the form} \\ \qquad A_{i_1} + A_{i_2} + \cdots + A_{i_k}, \ \Sigma(i_r+1) = \ell+1. \end{array} \right\}$$

$\uparrow$ for a connected component with Dynkin diagram of type A_{i_1}.

proof sketch. The i-cycle is a coxeter element of some Weyl subgp of type A_{i-1}.
Disjoint cycles operates on mutually orthogonal subspaces of V.

Theorem 9. (type B/C) Let W be a Weyl group of type C_ℓ. There is a one-to-one correspondence

$$\{\text{conjugacy classes of } W\} \leftrightarrow \left\{ \begin{array}{l} \text{admissible diagrams of the form} \\ \qquad A_{i_1} + A_{i_2} + \cdots + C_{j_1} + C_{j_2} + \cdots, \ \Sigma \underbrace{i_r+1}_{\lambda_r} + \Sigma \underbrace{j_r}_{\mu_r} = \ell. \end{array} \right.$$

proof sketch. A cycle $(k_1 k_2 \cdots k_r)$ means

$$e_{k_1} \to \pm e_{k_2} \to \cdots \to \pm e_{k_r} \to \pm e_{k_1}.$$

We say a cycle is <u>positive</u> if $w^r(e_{k_1}) = e_{k_1}$ and <u>negative</u> if $w^r(e_{k_1}) = -e_{k_1}$.
It is easy to see two elements in W are conjugate iff they have the same signed cycle type.

A positive i-cycle, <u>denoted $[i]$</u>, is a Coxeter element of a Weyl subgp of type A_{i-1}.
A negative i-cycle, <u>denoted $[\bar{i}]$</u>, is a Coxeter element of a Weyl subgp of type C_i.

(Note that a negative i-cycle

$$e_1 \to e_2 \to \cdots \to e_i \to -e_1 \to -e_2 \to \cdots \to -e_i \to e_1$$

can be expressed as product

$$(12) \cdot (23) \cdots (i\text{-}1, i) \cdot w_i$$

where $w_i(e_i) = -e_i$, $w_i(e_j) = e_j$ for $j \neq i$.
These factors form a complete set of simple reflections of Weyl subgp of type C_i operating on $e_1, \dots, e_i$, so negative i-cycles are Coxeter elements)

Disjoint cycles operates on mutually orthogonal subspaces of V.

Defining two partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ recover Young's results that

$$\{\text{conjugacy classes of } W(C_\ell)\} \overset{\text{bij}}{\longleftrightarrow} \{\text{pairs of partitions } (\lambda, \mu) \text{ with } |\lambda| + |\mu| = 1\}.$$

Theorem 10 (type D).

(i). An element of $W(C_\ell)$ lies in $W(D_\ell)$ iff it has an even number of negative terms in its signed cycle type. iff it changes signs of even number of basis vectors e_i.

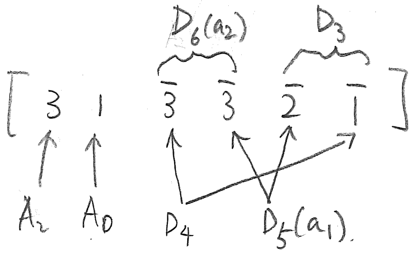
(ii). Two elements in $W(D_\ell)$ are conjugate iff they have same signed cycle type, except that if all cycles are even and positive, there are two conjugacy classes.

(iii). A positive i-cycle $[i]$ is represented by admissible diagram A_{i-1}.

A pair of negative cycles $[\bar{i}\,\bar{j}]$ with $i \geq j$ is represented by admissible diagram $\begin{cases} D_{i+1} & \text{if } j=1 \\ D_{i+j}(a_{j-1}) & \text{if } j>1. \end{cases}$

The admissible diagram representing any other class is obtained by splitting the signed cycle-type into positive cycles and pairs of negative cycles and the taking the union of corresp. admissible diagrams.

Eg. The conj. class with signed cycle-type $[3\,1\,\bar{3}\,\bar{3}\,\bar{2}\,\bar{1}]$ may be represented by graph $A_2 + D_3 + D_6(a_2)$ or $A_2 + D_4 + D_5(a_1)$.

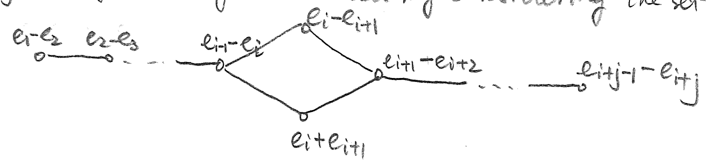


Proof sketch. (i). A positive cycle changes sign of an even number of e_i's.
$\cdots$ negative — — — — odd

(ii). Two elements having same signed cycle-type are $W(C_\ell)$-conj. We want to make sure they are $W(D_\ell)$-conj. Write them explicitly and do some computations.

- positive i-cycle is a Coxeter element of a Weyl subgp of type A_{i-1}.
- The pair of negative i-cycle
  - $[\bar{i}\,\bar{1}] = (12)(23)\cdots(i\!-\!1\ i)(i\ i\!+\!1)\,w_{i+1}$, where $w_{i+1}(e_i) = -e_{i+1}$. $w_{i+1}(e_{i+1}) = -e_i$. $w_{i+1}(e_j) = e_j$ for $j \neq i$.

    These elements form a complete set of simple reflections of $W(D_{i+1})$, so a Coxeter element.
  - $[\bar{i}\,\bar{j}]$, $i \geq j \geq 2$ may be obtained by considering the set of roots



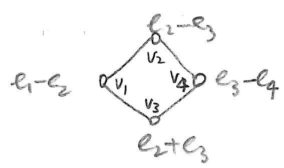
Dividing these roots in the only way possible into S_1, S_2 each mutually orthogonal.

$$\prod_{\alpha \in S_1} s_\alpha \cdot \prod_{\alpha \in S_2} s_\alpha$$

We obtain an element of signed cycle type $[\bar{i}\,\bar{j}]$. Thus can be represented by $D_{i+j}(a_{j-1})$.

Eg. $[\bar{2},\bar{2}]$ in D_4, say $(\overline{12})(\overline{43})$.

Consider



Let $S_1 = \{e_1-e_2, e_3-e_4\}$
$S_2 = \{e_2-e_3, e_2+e_3\}$.

Then $\underset{(12)}{\underset{\parallel}{s_{e_1-e_2}}}\ \underset{(34)}{\underset{\parallel}{s_{e_3-e_4}}}\ \underset{(23)}{\underset{\parallel}{s_{e_2-e_3}}}\ \underset{W_2:\ e_2\mapsto -e_3,\ e_3\mapsto -e_2}{\underset{\parallel}{s_{e_2+e_3}}} = (\overline{12})(\overline{43})$

So its admissible graph is exactly the graph above.

There are more than one graph representing a conjugacy class and more than one conjugacy classes represented by a given graph. The former is because signed cycle type decomposition (positive and pairs of negative) is not unique. The latter is because positive even corresp. to 2 classes and also because of isomorphisms $\underset{[\bar{1}\bar{1}]=[2\,2]}{D_2 = A_1+A_1}$, $\underset{[\bar{2}\bar{1}]=[4]}{D_3 = A_3}$.

Note that so far every conjugacy class in groups of type ABCD can be represented by admissible diagrams. Hence $W_0 = W$ for type ABCD.

For exceptional types no cycle-type description is available, so we need to work on roots.

Denote admissible diagram of an element of type A_i expressible as product of reflections correspondent to long roots (resp. short roots) as $\underline{A_i \text{ (resp. } \tilde{A}_i)}$.

Prop 11. For each of pairs of Γ_1, Γ_2 in table of lemma 26, P32, there exist elements $w_1, w_2 \in W_0$ s.t. w_i has Γ_i as admissible graphs, $i=1,2$, but w_1, w_2 are not conjugate in W.

For each of the graphs Γ in the table of lemma 27, P33, there exist $w_1, w_2 \in W_0$ corresponding to Γ but w_1 and w_2 are not conjugate in W.

proof sketch. Conjugated elements of W must fix/invert the same number of long roots and the same number of short roots, as if w fixes/invert α, $w'ww'^{-1}$ fixes/invert $w'(\alpha)$. Both α and $w'(\alpha)$ are long/short roots.

Def 4. Let W be a Weyl group of an exceptional type, with root system Φ.

1). Obtain all admissible diagrams associated with W by algorithm under theorem 6.

2). Calculate characteristic polynomial of all Γ's in 1). Note that different Γ's may have same characteristic polynomial.

3). For each characteristic polynomial in 2), choose a Γ giving rise to it, except in prop 11, choose both graphs.

Define $\underline{\mathcal{A}}$ to be the admissible diagrams chosen.

4). For each $\Gamma \in \mathcal{A}$, choose one conj. giving rise to it, except in prop 11, choose both classes.

Define $\underline{\mathcal{C}_e}$ to be the conjugacy classes chosen.

By a counting argument, we shall show $\mathcal{C}_e$ is the full set of conjugacy classes in W.

Let $\Gamma \in \mathcal{A}$, S the set of roots corresp. to nodes of Γ.
Let Φ_1 be the smallest root subsystem of Φ containing S., $U_1 = \text{span}\,\Phi_1$., W_1: Weyl subgroup of W for Φ_1
Let Φ_2 be roots in Φ orthogonal to U_1, $V_2 = \text{span}\,\Phi_2$. W_2: Weyl subgroup of W for Φ_2.
Then $\Phi_1 \cup \Phi_2$ is also a root subsystem with Weyl group $W_1 \times W_2$.

Prop 12. $W_1 \times W_2 \triangleleft N_W(W_1)$, the normalizer of W_1.

$N_W(W_1) / W_1 \times W_2 \cong$ the group of symmetries of Dynkin diagram of W_1 induced by transformations by elements of W.

proof sketch.. For $w \in N_W(W_1)$, $w s_\alpha w^{-1} = s_{w(\alpha)} \in W_1$. $w(\Phi_1) = \Phi_1$.

- $N_W(W_1) = W_1 \rtimes N_W(\Pi_1)$ where Π_1 is a simple root system of Φ_1, $N_W(\Pi_1) = \{w \mid w(\Pi_1) = \Pi_1\}$.
- $C_W(\Pi_1) = \{w \in W \mid w(\alpha) = \alpha \text{ for all } \alpha \in \Pi_1\} \cong W_2$.
- $N_W(W_1)/W_1 \times W_2 \cong N_W(\Pi_1)/C_W(\Pi_1) =$ group of symmetries of Π_1 induced by elements of W.

This can calculate the number of conjugates of W_1 in $W = |W : N_W(W_1)|$ if the induced symmetries of Dynkin diagram of W_1 is known.

Prop 13. Let $w \in W_0$ expressible as a product of reflection corresp. roots in S.
The number of conjugates of $w \in W$ is $\frac{abc}{d}$ where

a = # elements in the conj. class of W_1 containing w. Also denoted as $|ccl_{W_1}(w)|$.
b = # conjugacy classes of W_1 contained in the conj. class of W containing w.
c = # conjugates of W_1 in W. $(= |W : N_W(W_1)|$ as above$)$.
d = # subgps conjugate to W_1 in W containing W.

proof sketch: clear.

We omit the calculations of a, b, c, d P_{35}–P_{45}, except an algorithm to calculate the size of centralizers $|C_W(w)|$.

For suitable elements including the one in table of P_{37},

$$|C_W(w)| = \text{product of } d_i \text{ not divisible by order of } w.$$

where $d_1, d_2, \cdots, d_\ell$ be the degrees of basic polynomial invariants (c.f. Humphreys Reflection groups §3) of W. $|W| = d_1 \cdots d_\ell$.

Theorem 14. $\sum_{C \in \mathcal{C}} |C| = |W|$.

Corollary 15. (i) $\mathcal{C}$ is the complete set of conjugacy classes of W

(ii). $W_0 = W$, i.e., every element is a product of two involutions $w = w_1 w_2$.
$\Rightarrow$ every element is contained in some dihedral group $\langle w_1, w_2 \rangle$.
$\Rightarrow$ Each element $w \in W$ there is an involution $i \in W$ s.t. $iwi = w^{-1}$. (take $i = w_1$).
$\Rightarrow$ Every element of W is conjugate to its inverse.

(iii). Every conjugacy classes of W is associated with some admissible diagram in $\mathcal{A}$.

(iv). Every admissible diagram in $\mathcal{A}$ correspond to one conjugacy class, except the graphs in prop 11, corresp. to two conj. classes.

Applications.

Prop 16. • There is a one-to-one correspondence

$$\{\text{maximal tori in } G(\mathbb{F}_q)\}_{/conj} \leftrightarrow \{\text{conj. of } W, \text{ the weyl group of } G\}.$$
$$T_w(q). \longleftarrow w$$

• The order of $T_w(q) = f(q)$, $f(t)$ = characteristic polynomial of $w \curvearrowright V$.

• $|N_W(q) : T_w(q)| = |C_W(w)|$, N_W is the normalizer of T_w in G.

Another application is that the admissible diagrams can be used to parametrize nilpotent orbit in $\mathfrak{g}$. The connection between conjugacy class and nilpotent orbit is studied later by Kazhdan-Lusztig, Lusztig.