# MATH4240: Stochastic Processes Tutorial 9

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# Uniqueness of Stationary Distribution

Recall that

**Theorem 0.** Let $P$ be a stochastic $n \times n$ matrix over a finite state space $\mathcal{S}$. If $P$ satisfies the following assumptions:

**Assumption 1.** The left eigenvector w.r.t. 1 can be chosen to have all nonnegative entries.

**Assumption 2.** The eigenvalue 1 is a simple root of the characteristic polynomial of $P$.

**Assumption 3.** Except 1, all other eigenvalues have moduli less than 1.

Then, the chain has a unique stationary distribution $\pi$ and

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{pmatrix}$$

# Convergence to the stationary distribution: periodic case

This theorem tells is that the stationary distribution $\pi$ exists (actually implies the aperiodicity) and each row of the limit matrix $\lim_{k\to\infty} P^k$ is $\pi$.

Moreover, a basic theorem says that (see textbook, aperiodic case in Theorem 7 on p.73):

**Theorem 1.** If a MC is irreducible, positive recurrent and aperiodic, then the stationary distribution $\pi$ uniquely exists and

$$\lim_{k\to\infty} P^k(x,y) = \pi(y) = 1/m_y, \tag{1}$$

where $m_y = E_y(T_y)$ is the mean return time to $y$.

Also in previous tutorial, we learnt a method to calculate the limit matrix $\lim_{k\to\infty} P^k$ when the chain is reducible and each block of an irreducible closed set satisfies the 3 assumptions. So all the chains we have discussed are still aperiodic.

# Convergence to the stationary distribution: periodic case

Until now we know nothing about the periodic case. Actually in periodic case, $\lim_{k\to\infty} P^k$ does not exist even if the chain has a unique stationary distribution. But for the limit of subsequence, the following theorem is really crucial:

**Theorem 2.** Let $X_n$, $n \geq 0$, be an irreducible positive recurrent MC having stationary distribution $\pi$. If the chain is periodic with period $d$, then for any $x, y \in \mathcal{S}$, there is an integer $r$, $0 \leq r < d$, such that $P^n(x, y) > 0$ only if $n = md + r$ for some $m \in \mathbb{N}$, and

$$\lim_{m\to\infty} P^{md+r}(x, y) = d\pi(y). \tag{2}$$

**Proof.** The following 3 steps complete the proof.

**Step 1.** Consider the case of $x = y$.
For any $x \in \mathcal{S}$, by the definition of period ($d = g.c.d.\{n \geq 1 : P^n(x,x) > 0\}$),

$$P^n(x,x) > 0 \quad \text{only if} \quad n = md \ \text{ for some } m \in \mathbb{N}.$$

Let $Y_m = X_{md}$, $m \geq 0$. Then follows (1) we have a decomposition of $\mathcal{S}$. Suppose that $x$ belongs to some irreducible closed set $\mathcal{C}_i$.
Let $X_0 = Y_0 = x$, then the mean return time to $x$ with respect to $Y_m$ is $m_x/d$, where $m_x$ is the mean return time to $x$ with respect to $X_n$. If we restrict the chain $Y_m$ to a smaller state space $\mathcal{C}_i$, then the new MC is irreducible, positive recurrent and aperiodic. By Theorem 1, the new chain has a unique stationary distribution $\pi_i$ and

$$\lim_{m\to\infty} P^{md}(x,x) = \lim_{m\to\infty} Q^m(x,x) = \pi_i(x) = \frac{1}{m_x/d} = d/m_x = d\pi(x).$$

# Convergence to the stationary distribution: periodic case

**Step 2.** Find $r$ for general $x, y$.
Let $x, y \in \mathcal{S}$, and $r_1 = \min\{n \geq 1 : P^n(x, y) > 0\}$.
By irreducibility, we can choose $n_1$ such that $P^{n_1}(y, x) > 0$. Then

$$P^{r_1+n_1}(x, x) \geq P^{r_1}(x, y)P^{n_1}(y, x) > 0$$

which implies that $d$ is a divisor of $r_1 + n_1$. If $n$ satisfies $P^n(x, y) > 0$, then

$$P^{n+n_1}(x, x) \geq P^n(x, y)P^{n_1}(y, x) > 0$$

so that $d$ is also a divisor of $n + n_1$. As a result, $d$ is a divisor of $n - r_1$.
Let $r_1 = m_1 d + r$, where $0 \leq r < d$, then

$$P^n(x, y) > 0 \quad \text{only if} \quad n = md + r \ \text{ for some } m \in \mathbb{N}.$$

**Remark.** One can check that for $x \in \mathcal{C}_i$ and $y \in \mathcal{C}_j$,

$$r = \begin{cases} j - i, & \text{if } i \leq j, \\ j + d - i, & \text{if } i > j. \end{cases}$$

**Step 3.** Prove formula (2).
Now we can write

$$\begin{aligned} P^{md+r}(x,y) &= \sum_{j=1}^{md+r} P_x(T_y = j) P^{md+r-j}(y,y) \\ &= \sum_{k=0}^{m} P_x(T_y = kd + r) P^{(m-k)d}(y,y). \end{aligned}$$

# Convergence to the stationary distribution: periodic case

Apply the bounded convergence theorem to

$$a_m(k) = \begin{cases} P^{(m-k)d}(y,y), & 0 \le k \le m, \\ 0. & k > m, \end{cases}$$

and $p_k = P_x(T_y = kd + r)$, then by (4),

$$\begin{aligned} \lim_{m\to\infty} P^{md+r}(x,y) &= \lim_{m\to\infty} \sum_{k=0}^{\infty} P_x(T_y = kd+r) P^{(m-k)d}(y,y) \\ &= \sum_{k=0}^{\infty} P_x(T_y = kd+r) \lim_{m\to\infty} P^{(m-k)d}(y,y) \\ &= \sum_{k=0}^{\infty} d\pi(y) P_x(T_y = kd+r) \\ &= d\pi(y). \end{aligned}$$

# Convergence to the stationary distribution: periodic case

**Example.** Consider the Ehrenfest chain with $d = 3$. The transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

# Convergence to the stationary distribution: periodic case

Obviously the period of the chain is 2. By direct calculation, the chain has a unique stationary distribution $\pi = (1/8, 3/8, 3/8, 1/8)$. Simply applying Theorem 2, we have

$$\lim_{k\to\infty} P^{2k} = \begin{pmatrix} 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \end{pmatrix}$$

and

$$\lim_{k\to\infty} P^{2k+1} = \begin{pmatrix} 0 & 3/4 & 0 & 1/4 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix}$$

# Alternative definition of Poisson processes

In general, we consider the process in continuous time $\{X_t\}_{t\geq 0}$ with state space $\mathcal{S}$. Similar to discrete-time processes (for example, Markov chains), we can discuss the following two properties:

**Markov property.** For any $0 \leq s_1 \leq \cdots s_n \leq s \leq t$ and any $x_1, \cdots, x_n, x, y \in \mathcal{S}$,

$$P(X_t = y \mid X_{s_1} = x_1, \cdots, X_{s_n} = x_n, X_s = x) = P(X_t = y \mid X_s = x).$$

**Time homogeneity.** For any $0 \leq s \leq t$ and any $x, y \in \mathcal{S}$,

$$P(X_t = y \mid X_s = x) = P(X_{t-s} = y \mid X_0 = x) = P_x(X_{t-s} = y).$$

If the process $X_t$ satisfies above two properties, one can define the *transition function* as follows

$$P_{xy}(t) = P_x(X_t = y), \quad t \geq 0,\ x, y \in \mathcal{S}.$$

# Alternative definition of Poisson processes

Here is alternative way to define Poisson processes:

**Definition.** Poisson process $\{X_t\}_{t\geq 0}$ with rate $\lambda > 0$ is a time homogeneous process satisfying Markov property with state space $\mathcal{S} = \mathbb{N}$, initial state $X_0 = 0$, and transition function

$$P_{xy}(t) = \begin{cases} \dfrac{(\lambda t)^{y-x} e^{-\lambda t}}{(y-x)!}, & 0 \leq x \leq y,\ t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

# Alternative definition of Poisson processes

Actually, above definition is equivalent to the one in the textbook. To verify this relation, it suffices to prove that $X_t$ satisfies the following three properties (you will see that this way is more convenient and effective in many proofs):

(i) (*Initial state*) $X_0 = 0$;
(ii) (*Stationary Poisson increments*) $X_t - X_s$ has a Poisson distribution with parameter $\lambda(t-s)$ for $0 \leq s \leq t$.
(iii) (*Indenpendent increments*) $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$.

# Alternative definition of Poisson processes

**Proof.** (i) is directly given in definition.
(ii) By formula (3), one can see

$$P_{xy}(t) = P_{0,y-x}(t), \quad t \geq 0. \tag{4}$$

Hence for $0 \leq s \leq t$ and $y \geq 0$,

$$\begin{aligned}
P(X_t - X_s = y) &= \sum_{x=0}^{\infty} P(X_s = x, X_t = x + y) \\
&= \sum_{x=0}^{\infty} P(X_s = x) P(X_t = x + y \mid X_s = x) \\
&= \sum_{x=0}^{\infty} P(X_s = x) P_{x,x+y}(t - s) \quad \text{(by time homogeneity)} \\
&= \sum_{x=0}^{\infty} P(X_s = x) P_{0,y}(t - s) \qquad \text{(by (4))} \\
&= P_{0,y}(t - s) \qquad (5)
\end{aligned}$$

# Alternative definition of Poisson processes

(iii) For $0 \le t_1 \le t_2 \le \cdots \le t_n$ and $z_1, z_2, \cdots, z_{n-1} \in \mathbb{N}$,

$$\begin{aligned}
& P(X_{t_2} - X_{t_1} = z_1, X_{t_3} - X_{t_2} = z_2, \cdots, X_{t_n} - X_{t_{n-1}} = z_{n-1}) \\
= & \sum_{x=0}^{\infty} P(X_{t_1} = x) P(X_{t_2} - X_{t_1} = z_1, X_{t_3} - X_{t_2} = z_2, \cdots, \\
& X_{t_n} - X_{t_{n-1}} = z_{n-1} \mid X_{t_1} = x) \\
= & \sum_{x=0}^{\infty} P(X_{t_1} = x) P_{x, x+z_1}(t_2 - t_1) P_{x+z_1, x+z_1+z_2}(t_3 - t_2) \cdots \\
& P_{x+z_1+\cdots+z_{n-2}, x+z_1+\cdots+z_{n-1}}(t_n - t_{n-1}) \quad \text{(by Markov property and time}\ \\
= & \sum_{x=0}^{\infty} P(X_{t_1} = x) P_{0,z_1}(t_2 - t_1) P_{0,z_2}(t_3 - t_2) \cdots P_{0,z_{n-1}}(t_n - t_{n-1}) \quad \text{(by (}\ \\
= & P_{0,z_1}(t_2 - t_1) P_{0,z_2}(t_3 - t_2) \cdots P_{0,z_{n-1}}(t_n - t_{n-1}) \\
= & P(X_{t_2} - X_{t_1} = z_1) P(X_{t_3} - X_{t_2} = z_2) \cdots P(X_{t_n} - X_{t_{n-1}} = z_{n-1}).
\end{aligned}$$

The last step follows from formula (5).

# Sum of independent Poisson processes

Two continuous-time process $\{X_1(s)\}_{s\geq 0}$ and $\{X_2(t)\}_{t\geq 0}$ are said to be *independent* if for any time instants $s, t \geq 0$, random variables $X_1(s)$ and $X_2(t)$ are independent.

**Theorem.** Suppose that $X_1(t)$ and $X_2(t)$ are independent Poisson processes with rates $\lambda$ and $\mu$. Then $X_1(t) + X_2(t)$ is a Poisson process with rate $\lambda + \mu$.

**Proof.** The conclusion holds if and only if the process $X_1(t) + X_2(t)$ starts with initial state 0 and has the stationary and independent increments with certain Poisson distributions. The initial state is $X_1(0) + X_2(0) = 0$ clearly. Since both $X_1(t)$ and $X_2(t)$ have independent increments, so does $X_1(t) + X_2(t)$.

# Sum of independent Poisson processes

For time instants $s, t \geq 0$, let $Y = X_1(t+s) - X_1(s)$ and $Z = X_2(t+s) - X_2(s)$ be the increments. Then

$$\begin{aligned} P(Y+Z=n) &= \sum_{m=0}^{n} P(Y=m)P(Z=n-m) \\ &= \sum_{m=0}^{n} e^{-\lambda t}\frac{(\lambda t)^m}{m!} \cdot e^{-\mu t}\frac{(\mu t)^{n-m}}{(n-m)!} \\ &= \frac{e^{-(\lambda+\mu)t}}{n!}\sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(\lambda t)^m(\mu t)^{n-m} \\ &= e^{-(\lambda+\mu)t}\frac{\big((\lambda+\mu)t\big)^n}{n!}. \end{aligned}$$

Hence the increment $X_1(t+s) + X_2(t+s) - X_1(s) - X_2(s) \sim \text{Poi}((\lambda+\mu)t)$.

**Remark.** One can generalize above theorem to the sum of $n$ independent Poisson processes by induction.