# MATH4240: Stochastic Processes Tutorial 11

WANG, Jixin

The Chinese University of Hong Kong

*jxwang@math.cuhk.edu.hk*

17 April, 2021

# Supplementary Topic: Poisson races

**Example 2.** Consider a bank with two tellers. Three people, Alice, Betty, and Carol enter the bank at almost the same time and in that order. Alice and Betty go directly into service while Carol waits for the first available teller. Suppose that the two tellers have exponential service times with means 3 and 6 minutes.

**(a)** What is the expected total amount of time for Carol to complete her business?

**(b)** What is the expected total time until the last of the three customers leaves?

**(c)** What is the probability Carol is the last one to leave?

We have two ways to solve these problems. The first solution is to solve it directly while the second solution is an application of Theorems 1 & 2. (It is more interesting!)

**Solution 1.** Let $A, B, C$ be the serving time of Alice, Betty, and Carol respectively. Then $A \sim \text{Exp}(1/3)$ and $B \sim \text{Exp}(1/6)$. Moreover, we know

$$C \sim \begin{cases} A \sim \text{Exp}(1/3), & \text{if } A \le B, \\ B \sim \text{Exp}(1/6), & \text{if } A > B. \end{cases}$$

**(a)** The waiting time for Carol is $\min\{A, B\}$. Hence the expected total amount of time for Carol to complete is

$$\begin{aligned} E(\min\{A,B\} + C) &= E(\min\{A,B\}) + EC \\ &= E(\min\{A,B\}) + E(A)P(A \le B) + E(B)P(A > B) \\ &= \frac{1}{1/3+1/6} + 3 \cdot \frac{1/3}{1/3+1/6} + 6 \cdot \frac{1/6}{1/3+1/6} \\ &= 2 + 2 + 2 = 6 \text{ minutes.} \end{aligned}$$

**(b)** The expected total time until the last of the three customers leaves is

$$
\begin{aligned}
& E(\max\{A, B, \min\{A, B\} + C\}) \\
= \;& E(\min\{A, B\}) + E(\max\{A - \min\{A, B\}, B - \min\{A, B\}, C\}) \\
= \;& E(\min\{A, B\}) + E(\max\{B - A, C\} \mid B > A)P(B > A) + \\
& E(\max\{A - B, C\} \mid A > B)P(A > B) \\
& \text{(by the lack of memory property)} \\
= \;& E(\min\{A, B\}) + E(\max\{B, A\})P(B > A) + E(\max\{A, B\})P(A > B) \\
= \;& E(\min\{A, B\}) + E(\max\{B, A\}) \\
= \;& EA + EB \\
= \;& 9 \text{ minutes.}
\end{aligned}
$$

**(c)** The probability that Carol is the last one to leave is

$$
\begin{aligned}
& P(\max\{A, B, \min\{A, B\} + C\} = \min\{A, B\} + C) \\
= \ & P(\min\{A, B\} + C > \max\{A, B\}) \\
= \ & P(\min\{A, B\} + C > \max\{A, B\} \mid A < B)P(A < B) + \\
& P(\min\{A, B\} + C > \max\{A, B\} \mid A > B)P(A > B) \\
= \ & P(C > B - A \mid B > A)P(B > A) + P(C > A - B \mid A > B)P(A > B) \\
= \ & P(A > B)P(B > A) + P(B > A)P(A > B) \\
= \ & 2P(B > A)P(A > B) \\
= \ & 2 \cdot \frac{1/3}{1/3 + 1/6} \cdot \frac{1/6}{1/3 + 1/6} \\
= \ & \frac{4}{9}.
\end{aligned}
$$

**Solution 2.** Suppose that there are infinitely many customers in the bank waiting for service. Consider this infinite service sequence, then the numbers of customers served by each teller in $t$ minutes is a Possion process $X_i(t)$, $i = 1, 2$, with rate $1/3$ and $1/6$ respectively. Then, the sum $X(t) = X_1(t) + X_2(t)$ is a Poisson process with rate $1/3 + 1/6 = 1/2$. By Theorem 1 (conversed), each arrival in $X(t)$ has probability $(1/3)/(1/2) = 2/3$ to be in $X_1(t)$ (type-1), and probability $1 - 2/3 = 1/3$ to be in $X_2(t)$ (type-2).

Let $\tau_n$ be the waiting time for the $n$-th arrival in $X(t)$, and let $A_n$ be the event that the $n$-th arrival is of type-1. So $E(\tau_n) = 2n$ and $P(A_n) = 2/3$ for each $n \geq 1$. The first arrival in type-1 (or type-2) means that Alice (or Betty) completes her business. If $A_1$ (or $A_1^c$) happens, then the second arrival in type-1 (or type-2) means that Carol completes her business. Let $C$ be the serving time of Carol.

# Supplementary Topic: Poisson races

**(a)** Using the same argument as in Solution 1, the expected total amount of time for Carol to complete is $E(\tau_1 + C) =$
$E(\tau_1) + E(C|A_1)P(A_1) + E(C|A_1^c)P(A_1^c) = 2 + 3(2/3) + 6(1/3) = 6.$
**(b)** Let $Y$ be the extra serving time for the last person after time $\tau_2$. Note that no matter who is the first person to leave, the remaining two must be served by different tellers (*why?*). As the exponentially distributed random variable is memoryless, we have $E(Y \mid A_2) = 6$ (the mean serving time of the second teller) and $E(Y \mid A_2^c) = 3$ (the mean serving time of the first teller). Then the expected total time until the last of the three customers leaves is

$$E(\tau_2+Y) = E(\tau_2)+E(Y \mid A_2)P(A_2)+E(Y \mid A_2^c)P(A_2^c) = 4+6(2/3)+3(1/3)$$

**(c)** The probability that Carol is the last one to leave is

$$P(A_1 \cap A_2^c) + P(A_1^c \cap A_2) = (2/3)(1/3) + (1/3)(2/3) = 4/9.$$

# Set up a MJP with rate matrix

Now, we turn to consider the general Markov jump process (MJP) $\{X(t)\}$, $t \geq 0$ with state space $\mathcal{S}$. From the lecture we know that the transition function $P_{xy}(t) := P(X(t) = y \mid X(0) = x)$ is differentiable, and the Chapman-Kolmogorov equation (*C-K equation*) can be written as $P'(t) = P(t)D$ (*forward equation*) or $P'(t) = DP(t)$ (*backward equation*), where $D = [q_{xy}]_{x,y\in\mathcal{S}}$ is called the *rate marix*. In such a matrix, the entry $q_{xy}$ is the jump rate from $x$ to $y$ for $x \neq y$, while $q_{xx} = -q_x = -\sum_{y:y\neq x} q_{xy}$ is the negative leaving rate from $x$.

Note that once we get the rate matrix of a MJP, then the transition function $P_{xy}(t)$ can be found as the solution of the (backward or forward) C-K equation, that is,

$$P(t) = e^{tD} := \sum_{n=0}^{\infty} \frac{(tD)^n}{n!}.$$

Hence it is natural to ask: how to set up the rate matrix in a MJP? Let us see the following examples.

# Set up a MJP with rate matrix

**Example 3. Car checkup.** In a servicshop, the checkup of a car requires 3 operations in sequence: 1=engine overhaul, 2=air conditioner repair, 3=broken system replacement. After completing all the above, the car will leave (=4). Assume that this operation sequence is a Markov process, and the mean time in these operations are 1.2, 1.5, 2.5 hours respectively. For this model, we may ask: (a) what is the probability that after 4 hours the car is in step 3? (b) what is the probability that after 4 hours it is still in the shop? To solve these problems, we need to find the rate matrix $D$ first.

# Set up a MJP with rate matrix

Let $\mathcal{S} = \{1, 2, 3, 4\}$. Then 3 positive jump rates are given by

$$q_{12} = \frac{1}{1.2} = \frac{5}{6}, \quad q_{23} = \frac{1}{1.5} = \frac{2}{3}, \quad q_{34} = \frac{1}{2.5} = \frac{2}{5},$$

and other jump rates are 0. Hence we can write

$$D = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -5/6 & 5/6 & 0 & 0 \\ 0 & -2/3 & 2/3 & 0 \\ 0 & 0 & -2/5 & 2/5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Here 4 is obviously a absorbing state (so there is no jump or leaving at 4).

Moreover, the Markov matrix (see the lecture) is

$$Q = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Solving the C-K equation, $P(t) = e^{tD}$ (you may have a try!), then the probability in question (a) is the value $P_{13}(4)$; for (b), it is $1 - P_{14}(4)$.

# Set up a MJP with rate matrix

**Example 4. Barber shop.** Consider a barber shop with two barbers and two waiting chairs. Customers arrive at a rate of 5 per hour. Customers arriving to a fully occupied shop leave without being served. Assume that the service rate for each barber is 2 customers per hour. We may ask: (a) what is the probability to have 3 customers in the shop in the long run? (b) What is the expected time for the shop to be full, counting from the opening time? Here we only set up the rate matrix.

# Set up a MJP with rate matrix

Let $X(t)$ be the number of customers in the shop at time $t$. Then $\mathcal{S} = \{0, 1, 2, 3, 4\}$. For $x \in \{0, 1, 2, 3\}$, as the status means that the shop is not full, we have $q_{x,x+1} = 5$. For $x = 1$, only one barber serves, hence $q_{10} = 2$. For $x \in \{2, 3, 4\}$, both two barber are busy, thus $q_{x,x-1} = 2 + 2 = 4$. Therefore, we can write

$$D = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -5 & 5 & 0 & 0 & 0 \\ 2 & -7 & 5 & 0 & 0 \\ 0 & 4 & -9 & 5 & 0 \\ 0 & 0 & 4 & -9 & 5 \\ 0 & 0 & 0 & 4 & -4 \end{pmatrix} \end{array}.$$

# Set up a MJP with rate matrix

Moreover, the Markov matrix (see the lecture, $Q_{xy} = q_{xy}/q_x$ for $x \neq y$ and $q_x \neq 0$; $Q_{xy} = 0$ otherwise) is

$$Q = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \;\;0\;\; & \;\;1\;\; & \;\;2\;\; & \;\;3\;\; & \;\;4\;\; \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 2/7 & 0 & 5/7 & 0 & 0 \\ 0 & 4/9 & 0 & 5/9 & 0 \\ 0 & 0 & 4/9 & 0 & 5/9 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$