THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 Stochastic Processes, 2022-23 Term 2

Test 2

Answer all FOUR questions (Total: 100 points). Give adequate explanation and justification for all your computations and observations, and write your proofs and reasonings in a clear, rigorous and complete way as much as you can.

1. *(30pts)* Consider a branching chain $\{X_n\}_{n\geq 0}$ with the offspring distribution

$$p_0 = 0.5,\ p_1 = 0.1,\ p_3 = 0.4.$$

   (a) Determine the extinction probability ρ.

   (b) Let $X_0 = 1$. What is the probability that the population is extinct in the second generation ($X_2 = 0$), given that it did not die out in the first generation ($X_1 > 0$)?

   (c) What is the extinction probability if the offspring distribution is given by $p_i = (1-q)q^i$, $i = 0, 1, ...$, for some $0 < q < 1$? Explain your answer.

Solution.

1 (a). 15 pts. $\mu = \sum_{k=0}^{\infty} kp_k = p_1 + 3p_3 = 0.1 + 3 \times 0.4 = 1.3 > 1$. Then there exists a unique extinction probability $\rho \in (0,1)$. Set a function $\phi(t) = \sum_{k=0}^{\infty} p_k t^k = p_0 + p_1 t + p_3 t^3 = 0.5 + 0.1t + 0.4t^3$ for $0 < t < 1$, then ρ satisfies $\phi(\rho) = \rho$. Solve $\phi(\rho) = \rho$ for $0 < \rho < 1$ which gives $\rho = \frac{-1+\sqrt{6}}{2}$.

1 (b). 10 pts.

$$\begin{aligned}
P(X_2 = 0|X_1 > 0) &= \frac{P(X_2 = 0, X_1 > 0|X_0 = 1)}{P(X_1 > 0|X_0 = 1)} \\
&= \frac{P(X_2 = 0, X_1 = 1|X_0 = 1) + P(X_2 = 0, X_1 = 3|X_0 = 1)}{P(1,1) + P(1,3)} \\
&= \frac{P(X_2 = 0|X_1 = 1)P(1,1) + P(X_2 = 0|X_1 = 3)P(1,3)}{P(1,1) + P(1,3)} \\
&= \frac{P(1,0)P(1,1) + P(3,0)P(1,3)}{P(1,1) + P(1,3)} \\
&= \frac{P(1,0)P(1,1) + P(1,0)^3 P(1,3)}{P(1,1) + P(1,3)} \\
&= \frac{0.5 \times 0.1 + 0.5^3 \times 0.4}{0.1 + 0.4} \\
&= 0.2.
\end{aligned}$$

1 (c). 5 pts. By the geometric series and its first order derivative, we have

$$\begin{aligned}
\mu &= \sum_{i=0}^{\infty} i p_i \\
&= \sum_{i=0}^{\infty} i(1-q)q^i \\
&= (1-q)q\sum_{i=0}^{\infty} i q^{i-1} \\
&= (1-q)q\frac{1}{(1-q)^2} \\
&= \frac{q}{1-q}.
\end{aligned}$$

If $0 < q \le \frac{1}{2}$, then $\mu \le 1$ and $\rho = 1$. If $\frac{1}{2} < q < 1$, then $\mu > 1$ and there exists a unique extinction probability $\rho \in (0,1)$. Set

$$\phi(t) = \sum_{i=0}^{\infty} p_i t^i = \sum_{i=1}^{\infty} (1-q)q^i t^i = (1-q)\sum_{i=1}^{\infty} (qt)^i = \frac{1-q}{1-tq},$$

for $0 < t < 1$. Solve $\phi(\rho) = \rho$ for $0 < \rho < 1$ which gives $\rho = \frac{1-q}{q}$.

2. *(20pts)* Let $\{X_n\}_{n\ge 0}$ be a Markov chain on state space $\{1,2,3,4,5\}$ with transition matrix

$$P = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}.$$

(a) Is this chain irreducible? Is it aperiodic? Explain your answers.

(b) Find the stationary distribution if any.

(c) Suppose the chain starts in state 1. What is the expected number of steps until it is in state 1 again?

(d) Again, suppose $X_0 = 1$. What is the expected number of steps until the chain is in state 4?

Solution.

2 (a). 5 pts. Denote $i \to j$ if $P(i,j) > 0$, where P is the transition probability. Note that in this matrix

$$1 \to 2 \to 4 \to 1 \to 3 \to 5 \to 1,$$

the chain is irreducible. As $P(5,5) = \frac{1}{2} > 0$, the irreducible chain is aperiodic.

2 (b). 5 pts. It suffices to compute some distribution π such that $\pi P = \pi$, $\sum_i \pi(i) = 1$ and $\pi(i) \geq 0$ for all $i \in S$. To have $\pi P = \pi$, we need to solve $(P^T - I)\pi^T = 0$, where I is the identity matrix. Solve it, the solution set is

$$\{(\frac{5t}{7}, \frac{5t}{14}, \frac{5t}{14}, \frac{3t}{14}, t), t \in \mathbb{R}\}.$$

$\sum_i \pi(i) = 1$ gives $t = 14/37$. Set $\pi = (10/37, 5/37, 5/37, 3/37, 14/37)$. By direct computation, one verifies that $\sum_i \pi(i) = 1$ and $\pi(i) \geq 0$ for all $i \in S$. The Markov chain is irreducible and finite, so this π is the unique stationary distribution.

2 (c). 5 pts. The Markov chain is irreducible with finite state space, then

$$\mathbb{E}_1(T_1) = m_1 = \frac{1}{\pi(1)} = 3.7.$$

2 (d). 5 pts. As $P(4,1) = 1$, we have

$$\mathbb{E}_1(T_4) = \mathbb{E}_4(T_4) - 1 = \frac{1}{\pi(4)} - 1 = \frac{34}{3}.$$

3. *(30pts)* Let $\{X_n\}_{n\geq 0}$ be a Markov chain on state space $S = \{1,2,3,4,5,6,7\}$ with transition matrix

$$P = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

(a) Is the chain is irreducible? Explain your answer. If it is irreducible, further find the period of this chain.

(b) Is there any stationary distribution for the chain? Explain your answer. If yes, further find all possible stationary distributions and point out whether or not it is unique.

(c) Discuss the possible long time behavior of $P^n(x,y)$ for any x and y as $n \to \infty$ in term of subsequences of n.

Solution. 10 pts each.

3 (a). One can easily see that $1 \to 3 \to 6 \to 2 \to 5 \to 7 \to 1 \to 4 \to 6 \to 1$ and hence the chain is irreducible.

To show the chain is periodic with period 3, one may prove by induction that

$$P^{3n+1} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{1}$$

$$P^{3n+2} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \end{bmatrix} \end{array}, \tag{2}$$

$$P^{3n} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \end{bmatrix} \end{array}, \tag{3}$$

for $n = 0, 1, 2, \ldots,$ where $*$'s are some real numbers to be determined. This shows that the period

$$d := g.c.d\{n \in \mathbb{N} | P^n(0,0) \neq 0\},$$

is divisible by 3. Moreover, by direct computation, one has $P^3(0,0) = 71/192 \neq 0$. In particular $3|d$. Hence, we must have $d = 3$.

(b). It suffices to compute some distribution π such that $\pi P = \pi$, $\sum_i \pi(i) = 1$ and $\pi(i) \geq 0$ for all $i \in S$. To have $\pi P = \pi$, we need to solve $(P^T - I)\pi^T = 0$, where I is the identity matrix. Solve it, the solution set is $\{(116t/131, 363t/262, 237t/262, 29t/131, 150t/131, 333t, 262, t), t \in \mathbb{R}\}$. Sub $t = 262/1785$ and set $\pi = (232/1785, 121/595, 79/595, 58/1785, 20/119, 111/595, 262/1785)$. By direct computation, one verifies that $\sum_i \pi(i) = 1$ and $\pi(i) \geq 0$ for all $i \in S$. Hence, this π is our desired stationary distribution.

(c). The chain is irreducible and the state space is finite, in particular the chain is positive recurrent. If the chain is periodic with period 3, then there is an integer $r, 0 \leq r < 3$ such that $P^n(x, y) > 0$ only if $n = 3m + r$ for some $m \in \mathbb{N}$ and $\lim_{n\to\infty} P^{3n+r}(x, y) = 3\pi(y)$. Together with equations (1), (2) and (3), one has

$$\lim_{n\to\infty} P^{3n+1} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 237/595 & 58/595 & 60/119 & 0 & 0 \\ 0 & 0 & 237/595 & 58/595 & 60/119 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 335/595 & 262/595 \\ 0 & 0 & 0 & 0 & 0 & 335/595 & 262/595 \\ 0 & 0 & 0 & 0 & 0 & 335/595 & 262/595 \\ 232/595 & 363/595 & 0 & 0 & 0 & 0 & 0 \\ 232/595 & 363/595 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$\lim_{n\to\infty} P^{3n+2} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 335/595 & 262/595 \\ 0 & 0 & 0 & 0 & 0 & 335/595 & 262/595 \\ 232/595 & 363/595 & 0 & 0 & 0 & 0 & 0 \\ 232/595 & 363/595 & 0 & 0 & 0 & 0 & 0 \\ 232/595 & 363/595 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 237/595 & 58/595 & 60/119 & 0 & 0 \\ 0 & 0 & 237/595 & 58/595 & 60/119 & 0 & 0 \end{bmatrix} \end{array},$$

$$\lim_{n\to\infty} P^{3n} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 232/595 & 363/595 & 0 & 0 & 0 & 0 & 0 \\ 232/595 & 363/595 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 237/595 & 58/595 & 60/119 & 0 & 0 \\ 0 & 0 & 237/595 & 58/595 & 60/119 & 0 & 0 \\ 0 & 0 & 237/595 & 58/595 & 60/119 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 335/595 & 262/595 \\ 0 & 0 & 0 & 0 & 0 & 335/595 & 262/595 \end{bmatrix} \end{array}.$$

4. *(20pts)* Consider the Markov chain with state space $S = \{0, 1, 2, ...\}$ and transition probabilities:

$$P(x, x+1) = \frac{2}{3}; \quad P(x, 0) = \frac{1}{3}.$$

Show that the chain is positive recurrent and find the limiting distribution π.

Solution. For $\forall x, y \in S$,

$$\rho_{xy} > P(x,0)P^y(0,y) = P(x,0)P(0,1)^y = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^y > 0.$$

Then the Markov chain is irreducible. From $\pi P = \pi$ and $\sum_{x=0}^{\infty} \pi(x) = 1$, we have

$$\begin{aligned}
\pi(0) &= \sum_{x=0}^{\infty} \pi(x)P(x,0) = \frac{1}{3}\sum_{x=0}^{\infty} \pi(x) = \frac{1}{3}, \\
\pi(1) &= \pi(0)P(0,1) = \frac{1}{3} \times \frac{2}{3} \\
\pi(2) &= \pi(1)P(1,2) = \frac{1}{3} \times \left(\frac{2}{3}\right)^2, \\
&\cdots
\end{aligned}$$

By induction, $\pi(x) = (1-p)p^x$, $x \geq 0$. Then by geometric series, we can check that π satisfies both $\sum_{x=0}^{\infty} \pi(x) = 1$ and $\pi(x) = \sum_{y=0}^{\infty} \pi(y)P(y,x)$, $x \geq 0$. Hence the stationary distribution $\pi = (\frac{1}{3}, \frac{2}{9}, \frac{1}{3}\left(\frac{2}{3}\right)^2, \cdots)$. As the irreducible Markov chain has a stationary distribution, the chain is positive recurrent.

—THE END—