THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 - Stochastic Processes - 2022/23 Term 2

Homework 2
Due date: 6th February 2023

All questions are selected from the textbook. Please submit online through Blackboard your answers to Compulsory Part only. The late submission will not be accepted. Reference solutions to both parts will be provided after grading.

Compulsory Part
Chapter 1 (page 41): 1, 2, 3, 5, 10, 19, 20(a)

Optional Part
Chapter 1 (Page 41): 4, 6, 7, 8, 9, 11, 12

Compulsory Part:

1. Solution.

(a) $P(X_1 = 0 \mid X_0 = 0 \text{ and } X_2 = 0)$

$$= \frac{P(X_0 = 0, X_1 = 0, X_2 = 0)}{P(X_0 = 0, X_2 = 0)}$$

$$= \frac{P(X_0 = 0, X_1 = 0, X_2 = 0)}{P(X_0 = 0, X_1 = 0, X_2 = 0) + P(X_0 = 0, X_1 = 1, X_2 = 0)}$$

$$= \frac{\pi_0(0)(1-p)^2}{\pi_0(0)(1-p)^2 + \pi_0(0)pq}$$

$$= \frac{(1-p)^2}{(1-p)^2 + pq}\ .$$

(b) $P(X_1 \neq X_2)$

$$\begin{aligned}
&= P(X_0 = 0, X_1 \neq X_2) + P(X_0 = 1, X_1 \neq X_2) \\
&= P(X_0 = 0, X_1 = 0, X_2 = 1) + P(X_0 = 0, X_1 = 1, X_2 = 0) + \\
&\quad P(X_0 = 1, X_1 = 0, X_2 = 1) + P(X_0 = 1, X_1 = 1, X_2 = 0) \\
&= \pi_0(0)(1-p)p + \pi_0(0)pq + (1-\pi_0(0))qp + (1-\pi_0(0))(1-q)q \\
&= pq + \pi_0(0)(1-p)p + (1-\pi_0(0))(1-q)q.
\end{aligned}$$

2. Solution. Note that the state space is $\{0, 1, \cdots, d\}$. Recall that we want to find

$$P(x, y) = P(X_{n+1} = y \mid X_n = x).$$

If $x = 0$, then $y = 1$. In this case, it is easy to see that $P(x,y) = \begin{cases} 1, & \text{if } y = 1; \\ 0, & \text{if } y \neq 1. \end{cases}$

If $x = d$, then $y = d-1$. We can also see that $P(x,y) = \begin{cases} 1, & \text{if } y = d-1; \\ 0, & \text{if } y \neq d-1. \end{cases}$

If $x \in \{1, \cdots, d-1\}$, then exhausting four cases (i.e. black (or red) ball is chosen from box 1 and black (or red) ball is chosen from box 2), we can find that

$$P(x,y) = \begin{cases} \left(\frac{x}{d}\right)^2, & \text{if } y = x-1; \\ 2\left(\frac{x}{d}\right)\left(\frac{d-x}{d}\right), & \text{if } y = x; \\ \left(\frac{d-x}{d}\right)^2, & \text{if } y = x+1; \\ 0, & \text{otherwise.} \end{cases}$$

3. Solution. If $X_n = 0$, it is clear that $P(0, y) = f(y)$.
If $X_n \neq 0$, then

$$X_{n+1} = \begin{cases} X_n + \xi_{n+1} - 1, & \text{with probability } p; \\ X_n + \xi_{n+1}, & \text{with probability } 1 - p. \end{cases}$$

This means

$$\begin{aligned} P(x, y) &= P(X_{n+1} = y \mid X_n = x) \\ &= p \cdot P(\xi_{n+1} = y - x + 1) + (1 - p) \cdot P(\xi_{n+1} = y - x) \\ &= pf(y - x + 1) + (1 - p)f(y - x). \end{aligned}$$

Hence the transition function is

$$P(x, y) = \begin{cases} f(y), & x = 0; \\ pf(y - x + 1) + (1 - p)f(y - x), & x \neq 0. \end{cases}$$

5. Solution.

(a) For $n = 1$, $P_0(T_0 = 1) = P(0, 0) = 1 - p$.
For $n \geq 2$, $P(T_0 = n) = P(0, 1)P(1, 1)^{n-2}P(1, 0) = pq(1 - q)^{n-2}$.

(b) $P_0(T_1 = n) = P(0, 0)^{n-1}P(0, 1) = p(1 - p)^{n-1}$.

10. Solution. The transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) For $x = 0$, $P_0(T_0 = 1) = P_0(T_0 = 3) = 0$, $P_0(T_0 = 2) = P(0, 1)P(1, 0) = 1/3$.
For $x = 1$, $P_1(T_0 = 1) = P(1, 0) = 1/3$, $P_1(T_0 = 2) = 0$,
$P_1(T_0 = 3) = P(1, 2)P(2, 1)P(1, 0) = 4/27$.
For $x = 2$, $P_2(T_0 = 1) = P_2(T_0 = 3) = 0$, $P_2(T_0 = 2) = P(2, 1)P(1, 0) = 2/9$.
For $x = 3$, $P_3(T_0 = 1) = P_3(T_0 = 2) = 0$, $P_3(T_0 = 3) = P(3, 2)P(2, 1)P(1, 0) = 2/9$.

(b)

$$P^2 = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 7/9 & 0 & 2/9 \\ 2/9 & 0 & 7/9 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 0 & 7/9 & 0 & 2/9 \\ 7/27 & 0 & 20/27 & 0 \\ 0 & 20/27 & 0 & 7/27 \\ 2/9 & 0 & 7/9 & 0 \end{pmatrix}$$

(c) $\pi_1 = \pi_0 P = (1/12, 5/12, 5/12, 1/12)$, $\pi_2 = \pi_0 P^2 = (5/36, 13/36, 13/36, 5/36)$, $\pi_3 = \pi_0 P^3 = (13/108, 41/108, 41/108, 13/108)$.

19. Solution. **(a)** Note that $C_1 = \{1, 2, 3\}$ and $C_2 = \{4, 5, 6\}$ are two irreducible closed sets. Hence 0 is transient and $1, 2, 3, 4, 5, 6$ are recurrent.

(b) Clearly $\rho_{00} = P(0, 0) = 1/2$. We need to calculate two absorption probabilities $\rho_{C_1}(0)$ and $\rho_{C_2}(0)$. By one-step argument, we have $\rho_{C_1}(0) = (1/2)\rho_{C_1}(0) + 3/8$ and

$\rho_{C_2}(0) = (1/2)\rho_{C_2}(0) + 1/8$. Hence $\rho_{01} = \rho_{02} = \rho_{03} = \rho_{C_1}(0) = 3/4$ and $\rho_{04} = \rho_{05} = \rho_{06} = \rho_{C_2}(0) = 1/4$.

20. Solution. (a) There are two irreducible closed sets $C_1 = \{0, 1\}$ and $C_2 = \{2, 4\}$. Hence $3, 5$ are transient and $0, 1, 2, 4$ are recurrent.

Optional Part:

4. Solution.

(a).

$$\begin{aligned} P(C|\bigcup_i D_i) &= P(C \cap (\bigcup_i D_i))/P(\bigcup_i D_i) \\ &= (\sum_i P(C \cap D_i))/P(\bigcup_i D_i) \\ &= (\sum_i p * P(D_i))/P(\bigcup_i D_i) \\ &= p. \end{aligned}$$

(b).

$$\begin{aligned} P(\bigcup_i C_i|D) &= P((\bigcup_i C_i) \cap D)/P(D) \\ &= (\sum_i P(C_i \cap D))/P(D) \\ &= \sum_i P(C_i|D). \end{aligned}$$

(c).

$$\begin{aligned} \sum_i P(E_i|D)P(C|E_i \cap D) &= \sum_i P(E_i \cap C \cap D)/P(D) \\ &= P(C \cap D)/P(D) \\ &= P(C|D). \end{aligned}$$

(d). From $P(A|C_i) = P(B|C_i)$, one has $P(A \cap C_i) = P(B \cap C_i)$.

$$\begin{aligned} P(A|\bigcup_i C_i) &= P(A \cap (\bigcup_i C_i))/P(\bigcup_i C_i) \\ &= \sum_i P(A \cap C_i)/P(\bigcup_i C_i) \\ &= \sum_i P(B \cap C_i)/P(\bigcup_i C_i) \\ &= P(B|\bigcup_i C_i). \end{aligned}$$

6. Solution. In the Ehrenfest chain, the state space is $\{0, 1, \cdots, d\}$ and the transition function is

$$P(x, y) = \begin{cases} \frac{x}{d}, & y = x - 1; \\ \frac{d-x}{d}, & y = x + 1. \end{cases}$$

Hence

$$P(X_1 = 0) = P(X_0 = 1)P(1, 0) = \frac{d}{2^d} \cdot \frac{1}{d} = \frac{1}{2^d},$$

$$P(X_1 = d) = P(X_0 = d-1)P(d-1, d) = \frac{d}{2^d} \cdot \frac{1}{d} = \frac{1}{2^d},$$

and for $x \in \{1, \cdots, d-1\}$,

$$\begin{aligned}
P(X_1 = x) &= P(X_0 = x-1)P(x-1, x) + P(X_0 = x+1)P(x+1, x) \\
&= \frac{\binom{d}{x-1}}{2^d} \cdot \frac{d-x+1}{d} + \frac{\binom{d}{x+1}}{2^d} \cdot \frac{x+1}{d} \\
&= \frac{d!}{(d-x)!(x-1)!} \cdot \frac{1}{d \cdot 2^d} + \frac{d!}{(d-x-1)!x!} \cdot \frac{1}{d \cdot 2^d} \\
&= \frac{x\binom{d}{x} + (d-x)\binom{d}{x}}{d \cdot 2^d} \\
&= \frac{\binom{d}{x}}{2^d}.
\end{aligned}$$

Hence the distribution of X_1 is still the binomial distribution $B(d, 1/2)$.

7. Proof. From a direct derivation,

$$\begin{aligned}
P(X_0 = x_0 | X_1 = x_1, \cdots, X_n = x_n) &= \frac{P(X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n)}{P(X_1 = x_1, \cdots, X_n = x_n)} \\
&= \frac{P(X_0 = x_0)P(x_0, x_1) \cdots P(x_{n-1}, x_n)}{P(X_1 = x_1)P(x_1, x_2) \cdots P(x_{n-1}, x_n)} \\
&= \frac{P(X_0 = x_0)P(x_0, x_1)}{P(X_1 = x_1)} \\
&= \frac{P(X_0 = x_0, X_1 = x_1)}{P(X_1 = x_1)} \\
&= P(X_0 = x_0 | X_1 = x_1).
\end{aligned}$$

8. Proof. (a) Let $x_0 = x$ and $x_{n_0} = y$. Suppose that there exists $0 \le i < j \le n_0$ such that $x_i = x_j$. Then

$$P^{n_0+i-j}(x,y) > P(x,x_1)\cdots P(x_{i-1},x_i)P(x_j,x_{j+1})\cdots P(x_{n_0-1},y) > 0.$$

It is impossible since n_0 is the smallest positive integer such that $P^{n_0}(x,y) > 0$. Hence $x, x_1, \ldots, x_{n_0-1}, y$ are distinct.

(b) By (a), we have n_0+1 distinct states $x, x_1, \ldots, x_{n_0-1}, y$. Hence $n_0 + 1 \le d$, or equivalently, $n_0 \le d-1$.

(c) By (a) and (b),

$$\begin{aligned} P_x(T_y \le d-1) &\ge P_x(T_y = n_0) \\ &\ge P(x,x_1)P(x_1,x_2)\cdots P(x_{n_0-2},x_{n_0-1})P(x_{n_0-1},y) > 0. \end{aligned}$$

9. Proof. (a) By formula (29) in textbook,

$$\begin{aligned} P_x(T_y \le n+1) &= \sum_{k=0}^{n} P_x(T_y = k+1) \\ &= P_x(T_y = 1) + \sum_{k=1}^{n} P_x(T_y = k+1) \\ &= P(x,y) + \sum_{k=1}^{n}\left(\sum_{z\ne y} P(x,z)P_z(T_y = k)\right) \\ &= P(x,y) + \sum_{z\ne y} P(x,z)\sum_{k=1}^{n} P_z(T_y = k) \\ &= P(x,y) + \sum_{z\ne y} P(x,z)P_z(T_y \le n), \quad n \ge 0. \end{aligned}$$

(b) As in the definition $\rho_{xy} = P_x(T_y < \infty)$, we have

$$\begin{aligned} \rho_{xy} &= \lim_{n\to\infty} P_x(T_y \le n+1) \\ &= \lim_{n\to\infty}\left(P(x,y) + \sum_{z\ne y} P(x,z)P_z(T_y \le n)\right) \\ &= P(x,y) + \sum_{z\ne y} P(x,z)\left(\lim_{n\to\infty} P_z(T_y \le n)\right) \\ &= P(x,y) + \sum_{z\ne y} P(x,z)P_z(T_y < \infty) \\ &= P(x,y) + \sum_{z\ne y} P(x,z)\rho_{zy}. \end{aligned}$$

11. Solution. **(a)**

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/5 & 3/5 & 1/5 & 0 \\ 0 & 1/5 & 3/5 & 1/5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 8/25 & 2/5 & 6/25 & 1/25 \\ 1/25 & 6/25 & 2/5 & 8/25 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $\pi_1 = \pi_0 P = (1/10, 2/5, 2/5, 1/10)$, $\pi_2 = \pi_0 P^2 = (9/50, 8/25, 8/25, 9/50)$.

(c) For $x = 0$ or $x = 3$, $P_x(T_{\{0,3\}} = 1) = 1$, $P_x(T_{\{0,3\}} = 2) = 0$.

For $x = 1$ or $x = 2$, $P_x(T_{\{0,3\}} = 1) = 1/5$, $P_x(T_{\{0,3\}} = 2) = (3/5)(1/5) + (1/5)(1/5) = 4/25$.

12. Solution. **(a)**

$$P^2 = \begin{pmatrix} 1-p & 0 & p \\ 0 & 1 & 0 \\ 1-p & 0 & p \end{pmatrix}$$

(b) Clearly we have $P^4 = P^2 P^2 = P^2$.

(c) For $n = 2k, k \in \mathbb{N}_+$, $P^n = (P^2)^k = P^2$.

For $n = 2k+1, k \in \mathbb{N}_+$, $P^n = (P^2)^k P = P^2 P = P$.