Solution to Exercise 4

1. (a) When $\alpha = 1$, we can directly verify the convexity of $|x|$ by definition. When $\alpha > 1$, $f(x) = x^\alpha$ is twice-differentiable and $f''(x) = \alpha(\alpha-1)x^{\alpha-2} \geq 0$. Then the result follows from second-order condition.

(b) Let $f = \max(f_1, f_2)$. Since $\forall x, y \in \mathbb{R}^N, \lambda \in [0,1]$, we have

$$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda)f_i(y) \leq \lambda f(x) + (1-\lambda)f(y), i = 1, 2.$$

Therefore,

$$f(\lambda x+(1-\lambda)y) = \max(f_1(\lambda x+(1-\lambda)y), f_2(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y).$$

2. The fact that strict convexity implies convexity is obvious. To see that strong convexity implies strict convexity, note that strong convexity of f implies

$$f(\lambda x+(1-\lambda)y)-\mu\|\lambda x+(1-\lambda)y\|^2 \leq \lambda f(x)+(1-\lambda)f(y)-\lambda\mu\|x\|^2-(1-\lambda)\mu\|y\|^2.$$

But

$$\lambda\mu\|x\|^2 + (1-\lambda)\mu\|y\|^2 - \mu\|\lambda x + (1-\lambda)y\|^2 > 0, \forall x, y, x \neq y, \forall \lambda \in (0,1),$$

because $\|x\|^2$ is strictly convex . The claim follows.

To see that the converse statements are not true, observe that $f(x) = x$ is convex but not strictly convex and $f(x) = x^4$ is strictly convex but not strongly convex.

3. (a) We have

$$\nabla^2 f = \begin{pmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x^3} \end{pmatrix} = \frac{2}{x_2^3}\begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix}.$$

Since $\nabla^2 f \succeq 0, \forall (x_1, x_2) \in \mathbb{R} \times (0, \infty)$, $f\,(x_1, x_2)\, is convex on \mathbb{R} \times (0, \infty)$.

(b) Just note that the Hessian of f is

$$\nabla^2 f = \left(\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j}\right)_{1 \leq i,j \leq N} = P$$

and use second-order condition.